Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 8
Solutions

1. Consider the compact group S^1 (with the standard multiplication of complex numbers). Let $p\colon \mathbb{R} \to S^1$ be defined by

$$p(t) = e^{2\pi i t} = \cos(2\pi(t)) + i\sin(2\pi(t)).$$

Prove that the Haar integral for S^1 is given by the formula

$$\int_{S^1} f(x)dx = (R)\int_0^1 (f \circ p)(t)dt$$

for all continuous $f\colon S^1 \to \mathbb{R}$. Here the integral of the right side is the usual Riemann integral on reals.

Solution: It is enough to check that the mapping $I\colon Map(S^1, \mathbb{R}) \to \mathbb{R}$ defined by

$$I(f) = (R)\int_0^1 (f \circ p)(t)dt$$

satisfies all properties of the Haar integral. The linearity of I as well as the fact that $I(f) \geq 0$ if $f \geq 0$ are obvious.

$$I(1) = (R)\int_0^1 (1 \circ p)(t)dt = (R)\int_0^1 1dt = 1.$$

Finally suppose $y \in S^1$ is arbitrary. There exists $s \in [0, 1[$ such that $p(s) = y$. Now

$$I(R_y(f)) = (R)\int_0^1 f(p(t)y)dt = (R)\int_0^1 f(p(t)p(s))dt = (R)\int_0^1 f(p(t+s))dt,$$

since p is a group homomorphism. By the properties of the Riemann integral

$$(R)\int_0^1 f(p(t+s))dt = (R)\int_s^{1+s} f(p(t))dt = (R)\int_s^1 f(p(t))dt + (R)\int_1^{1+s} f(p(t))dt.$$

Since p is periodic with period 1 it follows that

$$(R)\int_1^{1+s} f(p(t))dt = (R)\int_0^s f(p(t-1))dt = (R)\int_0^s f(p(t))dt,$$

hence finally we obtain

$$I(R_y(f)) = (R)\int_s^1 f(p(t))dt + (R)\int_0^s f(p(t))dt = (R)\int_0^1 f(p(t))dt = I(f).$$

Since S^1 is abelian (or by exercise 7.4) this is enough.

2. Let $n \in \mathbb{N}$ and define $f\colon S^1 \to S^1$ by $f(z) = z^n$. Use the previous exercise to calculate the value of the Haar integral $\int f$ (where we think of f as mapping $f\colon S^1 \to \mathbb{R}^2$).

   Solution: If $n = 0$ f is a constant function 1, so $\int f = 1$. Otherwise
$$\int f = (\int_0^1 cos2\pi ntdt, \int_0^1 \sin 2\pi ntdt) = (0,0) = 0.$$

3. Suppose V is an n-dimensional vector space, $n \in \mathbb{N}$. Let $A\colon V \to \mathbb{R}^n$ be a linear isomorphism. Define **Euclidean topology** on V by requiring that A is a homeomorphism.
   a) Show that this topology is uniquely defined and does not depend on the choice of the isomorphism A.
   b) Suppose V, W are finite-dimensional vector spaces and $L\colon V \to W$ is a linear mapping. Show that L is continuous with respect to the Euclidean topologies on V and W.
   (Hint: prove b) first and apply it to the identity mapping to derive a)).
   c) Let $GL(V)$ be the group of all linear isomorphisms $L\colon V \to V$ (with respect to the composition of mappings) and let $v = \{v_1, \ldots, v_n\}$ be an (ordered) basis of V. Show that the mapping
$$\mu_v\colon GL(V) \to GL(n, \mathbb{R})$$
   defined by $\mu_v(A) = [A]_{v,v}$ (the matrix of A with respect to the basis v) is an isomorphism of groups and the topology co-induced by μ_v in the group $GL(V)$ does not depend on the choice of the basis of V. Show that $GL(V)$ is a topological group with respect to this topology (also referred to as the Euclidean topology of the group $GL(V)$).

   Solution: Suppose $A\colon V \overset{\cong}{\to} \mathbb{R}^n$, $B\colon W \overset{\cong}{\to} \mathbb{R}^m$ are linear isomorphisms and give V topology induced by A and W topology induced by B. Suppose $L\colon V \to W$ is linear. Then $L' = B \circ L \circ A^{-1}\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping between spaces $\mathbb{R}^n$, $\mathbb{R}^m$ and such a mapping is always continuous with respect to the standard topologies of $\mathbb{R}^n, \mathbb{R}^m$. Since A and B are now homeomorphisms, L is continuous. In particular b) is true.

   Now suppose V is given two topologies τ_1, τ_2 using different isomorphisms $A, A'\colon V \to \mathbb{R}^n$. Then $\mathrm{id}\colon (V, \tau_1) \to (V, \tau_2)$ and $\mathrm{id}^{-1} = \mathrm{id}\colon (V, \tau_2) \to (V, \tau_1)$ are linear, so are continuous by what we already proved. Hence $\tau_1 = \tau_2$.

   To prove c) first notice that the set $\{L\colon V \to V \mid L \text{ is linear }\} = W$ has the natural structure of finite-dimensional vector space, $\dim W = (\dim V)^2$. The mapping μ_v can be extended to a linear isomorphism
$$\mu_v\colon W \to M(n; \mathbb{R})$$
   which is defined by the same formula - $\mu_v(A) = [A]_{v,v}$. Now by the first part of this exercise the topology of W induced by μ_v does not depend on v, so the same is true for the relative topology on $GL(V)$.

4. Suppose G is a group and V is a finite-dimensional vector space. No topologies are considered (yet). A homomorphism of groups $\phi\colon G \to GL(V)$ is called **a linear representation** of G in V. A mapping $\Phi\colon G \times V \to V, \Phi(g,v) = gv$ is called **linear action** of G on V if it satisfies algebraic conditions of the action i.e.
i)$ev = v$ for all $v \in V$, e neutral element of G,
ii)$(gg')v = g(g'v)$ for $g, g' \in G, v \in V$
and also
iii)the mapping $\Phi_g\colon V \to V$ defined by $v \mapsto gv$ is linear for every $g \in G$.

a) Suppose $\phi\colon G \to GL(V)$ is a linear representation of G in V. Define $\hat{\phi}\colon G \times V \to V$ by $\hat{\phi}(g,v) = \phi(g)(v)$. Prove that the correspondence $\phi \to \hat{\phi}$ is a bijection from the set of all linear representations of G and the set of all linear actions of G on V.

b) Now suppose G is a topological group and equip V with its Euclidean topology. Prove that a linear representation ϕ of G in V is continuous (with respect to the Euclidean topology in $GL(V)$) if and only if the corresponding linear action $\hat{\phi}$ is continuous.

c) Suppose G is a topological group and V is a finite-dimensional vector space with the Euclidean topology. Suppose $\Phi\colon G \times V \to V, \Phi(g,v) = gv$ is a linear action of G on V as defined above (not assumed continuous). Deduce that the following conditions are equivalent:
i) Φ is continuous.
ii) For every fixed $v \in V$ a mapping $\Phi_v\colon G \to V$ defined by $\Phi_v(g) = gv$ is continuous.
iii) There exists a basis $\{v_1, \ldots, v_n\}$ of V such that Φ_{v_i} is continuous for all $i = 1, \ldots, n$.

Solution: a) Suppose $\phi\colon G \to GL(V)$ is a homomorphism. Then

$$\hat{\phi}(e,v) = \phi(e)(v) = \mathrm{id}(v) = v,$$

$$\hat{\phi}(g, \hat{\phi}(g',v)) = \phi(g)(\hat{\phi}(g',v)) = \phi_g(\phi_{g'}(v)) = \phi_g \circ \phi_{g'}(v) = \phi_{gg'}(v) = \hat{\phi}(gg',v),$$

since ϕ is homomorphism. Also for every $g \in G$ $\hat{\phi}_g = \phi(g)$ is linear.
Hence $\hat{\phi}$ is a linear action.

Conversely suppose $\Phi\colon G \times V \to V$ is a linear action. Then Φ_g is linear. Since

$$\Phi_g \circ \Phi_{g^{-1}} = \mathrm{id} = \Phi_{g^{-1}} \circ \Phi_g$$

Φ_g is in fact a linear isomorphism, i.e. an element of $GL(V)$. Hence we may define $\hat{\Phi}\colon G \to GL(V)$ by $\hat{\Phi}(g) = \Phi_g$. Using condition (ii) of action it is easy to see that $\hat{\Phi}$ is a homomorphism of groups. Also $\phi \mapsto \hat{\phi}$ and $\Phi \mapsto \hat{\Phi}$ are converse operations of each other. Hence they are both bijective correspondences.

Let us next prove c) first. Clearly i)$\Rightarrow$ ii) $\Rightarrow$ iii). Suppose iii) is true. let $\{v_1, \ldots, v_n\}$ be a basis for V such that Φ_{v_i} is continuous for all $i = 1, \ldots, n$.

Since $\{v_1, \ldots, v_n\}$ is a basis every vector $v \in V$ can be written in unique way as a linear combination of basis vectors,

$$v = \sum_{i=1}^{n} a_i(v) v_i,$$

where $a_i \colon V \to \mathbb{R}$ are continuous with respect to the Euclidean topology (since they correspond to projections for a suitable isomorphism $V \to \mathbb{R}^n$). Now

$$\Phi(g, v) = \Phi_g(v) = \Phi_g(\sum_{i=1}^{n} a_i(v) v_i) = \sum_{i=1}^{n} a_i(v) \Phi_g(v_i) = \sum_{i=1}^{n} a_i(v) \Phi_{v_i}(g),$$

since Φ_g is linear for all $g \in G$. Since algebraic operations, mappings a_i and Φ_{v_i} are continuous, it follows that Φ is continuous.

Now we can finally prove b). Suppose $\phi \colon G \to GL(V)$ is continuous and choose a basis $\{v_1, \ldots, v_n\}$ of V. Then $\phi' = \mu_v \circ \phi \colon G \to GL(n; \mathbb{R})$ is continuous and the entries $\phi_{ij}(g)$ of the matrix $\phi'(g)$ are exactly the coordinates of $\hat{\phi}_{v_i}(g)$. Since former are continuous, it follows that $\hat{\phi}_{v_i}$ is continuous for all i, which implies the continuity of $\hat{\phi}$ by what we already proved.

Also the converse is true - if Φ is a linear action, and $\{v_1, \ldots, v_n\}$ is some basis of V, then the coordinates of $\Phi_{v_i}(g)$ are exactly the entries of the matrix $\mu_v \circ \hat{\Phi}(g) \in GL(n; \mathbb{R})$. Hence the continuity of Φ implies the continuity of $\hat{\Phi}$.

5. Suppose G is a topological group, V is a finite-dimensional vector space and $\phi \colon G \to V$ is a continuous linear representation. A mapping $f \colon Map(G, \mathbb{R}) \to \mathbb{R}$ is called **matrix coefficient** of the representation ϕ is there exists $v \in V$ and linear $L \colon V \to \mathbb{R}$ such that

$$f(g) = L(\phi(g)(v))$$

for all $g \in G$. The vector subspace of $Map(G, \mathbb{R})$ spanned by all matrix coefficients of ϕ will be denoted $\mathcal{M}_\phi$.
a) Choose a basis in $v_1, \ldots, v_n$ in V and represent all linear mapping $\phi(g)$ in that basis as matrices:

$$\phi(g) = \begin{bmatrix} \phi(g)_{11} & \ldots \phi(g)_{1n} \\ \ldots & \ldots \\ \phi(g)_{n1} & \ldots \phi(g)_{nn} \end{bmatrix}$$

Prove that every mapping $\phi(g)_{ij}$ obtained from the corresponding coefficient of such matrix is a matrix coefficient of ϕ (which explains terminology).

b) Prove that every matrix representation of ϕ can be written as a linear combination of matrix coefficients $\phi(g)_{ij}$ defined as above. Conclude that $\mathcal{M}_\phi$ is finite-dimensional and in fact $\dim \mathcal{M}_\phi \leq (\dim V)^2$.

Solution: a) For every $j = 1, \ldots, n$ define linear mapping $L_j \colon V \to \mathbb{R}$ as follows. Suppose $v \in V$. Then there exists unique representation of v as a linear combination of vectors from the basis $v_1, \dots, v_n$,

$$x = a_1(x)v_1 + a_2(x)v_2 + \ldots + a_n(x)v_n.$$

Assert $L_j(x) = a_j(x)$. It is easy to verify that L_j is linear. Now

$$\phi_{ij}(g) = L_i(\phi(g)v_j),$$

so ϕ_{ij} is a matrix coefficient.

b) Let L_j be as above, $j = 1, \dots, n$. Then for every linear $L\colon V \to \mathbb{R}$ and $v \in V$ we have

$$L(v) = L(L_1(v)v_1+L_2(v)v_2+\ldots+L_n(v)v_n) = L_1(v)L(v_1)+L_2(v)L(v_2)+\ldots+L_n(v)L(v_n) =$$

$$= b_1L_1(v) + b_2L_2(v) + \ldots + b_nL_n(v),$$

where $b_i = L(v_i)$ does not depend on v. Hence

$$L = b_1L_1 + b_2L_2 + \ldots + b_nL_n$$

i.e. every linear $L\colon V \to \mathbb{R}$ can be written as a linear combination of mapping L_j. Hence if f is a matrix coefficient, $f(g) = L(\phi(g)v)$ for some L, v, we have

$$f(g) = (b_1L_1 + b_2L_2 + \ldots + b_nL_n)(\phi(g)(a_1v_1 + a_2v_2 + \ldots + a_nv_n) =$$

$$= \sum_{1\le i,j\le n} a_ib_jL_j(\phi(g)v_i) = \sum_{1\le i,j\le n} a_ib_j\phi_{ij}(g),$$

where $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$. Hence f belongs to the spanned by ϕ_{ij}, $i, j = 1, \dots, n$, where $n = \dim V$. In particular $\dim \mathcal{M}_\phi \le (\dim V)^2$.

6. Suppose f is a matrix coefficient of a continuous linear representation $\phi\colon G \to V$ and suppose $g \in G$. Show that R_gf and L_gf are also matrix coefficients of ϕ. Conclude that the vector subspace of $Map(G, \mathbb{R})$ spanned by the set

$$\{R_gf \mid g \in G\}$$

is finite-dimensional and the same is true for the vector subspace spanned by the set

$$\{L_gf \mid g \in G\}.$$

Solution: Suppose for all $g \in G$ $f(g) = L(\phi(g)v)$ for some fixed linear $L\colon V \to \mathbb{R}$ and $v \in V$. Then if $h \in H$

$$R_hf(g) = f(gh) = L(\phi(gh)v) = L(\phi(g)(\phi(h)v)) = L(\phi(g)v'),$$

where $v' = \phi(h)(v) \in V$. Hence R_hf is a matrix coefficient of ϕ.
Likewise

$$L_hf(g) = f(hg) = L(\phi(h)(\phi(g)v)) = L'(\phi(g)v),$$

where $L' = L \circ \phi(h)\colon V \to \mathbb{R}$ is linear. Hence L_hf is a matrix coefficient of ϕ.

Since the subspace spanned by all matrix coefficients of ϕ is finite-dimensional by the previous exercise, it follows that in particular its subspaces generated by $\{R_gf \mid g \in G\}$ or $\{L_gf \mid g \in G\}$ are finite-dimensional.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.