Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 7
12-16.03.2012

1. Suppose G is a topological group, $n \in \mathbb{N}$. Let

$$V = \{f : G \to \mathbb{R}^n \mid f \text{ is continuous}\}.$$

Then V is a vector space with addition and scalar multiplication defined pointwise,

$$(f + f')(x) = f(x) + f'(x),$$

$$(af)(x) = af(x).$$

Let $h \in G$ be fixed. Define $R_h, L_h : V \to V$ by

$$R_h(f)(g) = f(gh),$$

$$L_h(f)(g) = f(hg).$$

Prove that R_h and L_h are linear isomorphisms of V and we have the following identities

$$R_h \circ R_{h'} = R_{hh'},$$

$$L_h \circ L_{h'} = L_{h'h},$$

$$R_e = \mathrm{id} = L_e.$$

Solution: R_h is linear, since

$$R_h(af+bf')(g) = (af+bf')(gh) = af(gh)+bf'(gh) = aR_h(f)(g)+bR_h(f')(g) =$$

$$= (aR_h(f) + bR_h(f'))(g)$$

for all $g \in G, a, b \in \mathbb{R}, f, f' \in V$. Similarly one sees that each L_h is linear.
Next we see that if $h, h' \in G$, then

$$(R_h \circ R_{h'})(f) = R_h(R_{h'}(f))(g) = (R_{h'}f)(gh) = f((gh)h') = f(g(hh')) = (R_{hh'})(f),$$

$$(L_h \circ L_{h'})(f) = L_h(L_{h'}(f))(g) = (L_{h'}f)(hg) = f(h'(hg)) = f((h'h)g)) = (L_{h'h})(f),$$

hence $R_h \circ R_{h'} = R_{hh'}$, $L_h \circ L_{h'} = L_{h'h}$. Also

$$(R_e f)(g) = f(ge) = f(g) = f(eg) = (L_e f)(g),$$

so $R_e = \mathrm{id} = L_e$. It follows that

$$R_h \circ R_{h^{-1}} = R_{hh^{-1}} = R_e = \mathrm{id} = R_e = R_{h^{-1}h} = R_{h^{-1}} \circ R_h,$$

$$L_h \circ L_{h^{-1}} = L_{h^{-1}h} = L_e = \mathrm{id} = L_e = L_{hh^{-1}} = L_{h^{-1}} \circ L_h,$$

hence R_h is bijective, with $R_{h^{-1}}$ its inverse, and the same for L_h.

2. Suppose G is a topological group, $A \subset G$, $a \in A$, (X, d) metric space and $f\colon A \to X$ a mapping.

a) Prove that f is continuous at a if and only if for every $\varepsilon > 0$ there exists a neighbourhood U of the neutral element $e \in G$ such that

$$d(f(x), f(a)) < \varepsilon$$

for all $x \in A$ which satisfy $xa^{-1} \in U$.

b) Suppose A above is **compact**. Prove that $f\colon A \to X$ is continuous if and only if for every $\varepsilon > 0$ there exists a neighbourhood U of the neutral element $e \in G$ such that

$$d(f(x), f(y)) < \varepsilon$$

for all $x, y \in A$ which satisfy $xy^{-1} \in U$.

Solution: a)f is continuous at a if and only if for every $\varepsilon > 0$ there exists a neighbourhood V of a such that $x \in V$ implies

$$d(f(x), f(a)) < \varepsilon.$$

But since G is a topological group every neighbourhood V of a is of the form Ua for some neighbourhood U of e and vise versa. Since $x \in Ua$ if and only if $xa^{-1} \in U$, we are done.

b) For every $y \in A$ choose a neighbourhood U_y of e such that

$$d(f(x), f(y)) < \varepsilon/2$$

for all $x, y \in A$ which satisfy $xy^{-1} \in U_y$. Since G is a topological group for every $y \in A$ there exists a neighbourhood V_y of e such that $V_y^2 \subset U_y$. Open sets $V_y y$ form an open cover of A. Since A is compact there exist finitely many points $y_1, \ldots, y_n$ such that $A \subset V_{y_1} y_1 \cup V_{y_2} y_2 \ldots \cup V_{y_n} y_n$. Let

$$V = \cap_{i=1}^{n} V_{y_i}.$$

Then V is an open neighbourhood of e. Suppose $x, y \in A, x \in Vy$. Since $y \in A$, there exists y_i, $i = 1, \ldots, n$ such that $y \in V_{y_i} y_i$, hence

$$x \in V V_{y_i} y_i \subset V_{y_i} V_{y_i} y_i \subset U_{y_i} y_i.$$

By the definition of U_{y_i} this implies that

$$d(f(x), f(y_i)) < \varepsilon/2.$$

Also $y \in U_{y_i} y_i$, so $d(f(y), f(y_i)) < \varepsilon/2$. Combining these and using triangle-inequality we obtain

$$d(f(x), f(y)) \leq d(f(x), f(y_i)) + d(f(y), f(y_i)) < \varepsilon.$$

Remark: Similarly one can prove that if $f\colon A \to X$ is continuous, A compact subset of the group G and X metric space, then for every $\varepsilon > 0$ there exists a neighbourhood U of the neutral element $e \in G$ such that

$$d(f(x), f(y)) < \varepsilon$$

for all $x, y \in A$ which satisfy $x^{-1}y \in U$.
This version of this result will be convenient in the next exercise.

3. Suppose G is a compact topological space and let
$$V = \{f\colon G \to \mathbb{R}^n \mid f \text{ is continuous}\}$$
be as above. Define a norm $|\cdot|$ in the vector space V by
$$|f| = \sup\{|f(x)| \mid x \in G\}.$$
We consider V a metric space with metric $d(f, g) = |f - g|$ defined by this norm.

a) Suppose $f \in V$ and $\varepsilon > 0$. Prove that there exists a neighbourhood U of the neutral element $e \in G$ such that
$$d(R_g(f), R_h(f)) < \varepsilon$$
for all $g, h \in G$ that satisfy $gh^{-1} \in U$. (Hint: previous exercise.)

b) Prove that $\Phi\colon G \times V \to V$, $\Phi(g, f) = R_g(f)$ is a continuous action of G on V.

Solution: a) If we modify desired conclusion slightly and use the remark above we can easily prove the following. By the remark above there exists a neighbourhood U of the neutral element e such that
$$|f(x) - f(y)| < \varepsilon$$
when $x^{-1}y \in U$. Using that we see that if $g^{-1}h \in U$, then $(xg)^{-1}(xh) = g^{-1}h \in U$ for every $x \in G$, so
$$|R_g(x) - R_h(x)| = |f(xg) - f(xh)| < \varepsilon.$$
Hence we have shown that
$$d(R_g(f), R_h(f)) < \varepsilon$$
when $g^{-1}h \in U$. This is NOT the conclusion asked in the exercise but it would suffice to prove the claim of b), which is the main object of interest.

To prove the exact claim asked, we have to do more work. Let us first prove that if G is a topological group and U is a neighbourhood of e, then there exists a neighbourhood V of e such that $gVg^{-1} \subset U$ for all $g \in G$.
Consider the mapping $f\colon G \times G \to G$ defined by $f(g, h) = ghg^{-1}$. This is continuous mapping and $G \times \{e\} \subset f^{-1}U$. Since $f^{-1}U$ is open and $G, \{e\}$ are compact there exists (see e.g. Väisälä, Topologia II, Lemma 15.24) neighbourhood V of e such that $G \times V \subset f^{-1}U$. This V satisfies the condition required.

Now as above we first find a neighbourhood U of e such that
$$|f(x) - f(y)| < \varepsilon$$

when $xy^{-1} \in U$ and let V be as above. Suppose $gh^{-1} \in V$, then for every $g \in G$

$$(xg)(xh)^{-1} = x(gh^{-1})x^{-1} \in xVx^{-1} \subset U,$$

so

$$|R_g f(x) - R_h f(x)| = |f(xg) - f(xh)| < \varepsilon$$

whenever $gh^{-1} \in V$.

b) Let us prove the continuity of Φ at $(g, f) \in G \times V$. Let $\varepsilon > 0$ and choose a neighbourhood U of the neutral element e such that

$$d(R_g(f), R_h(f)) < \varepsilon/2$$

for all $h \in G$ that satisfy $gh^{-1} \in U$ i.e. $h \in U^{-1]}g$. Suppose $f' \in V$ is such that $d(f', f) < \varepsilon/2$. Then for any $h \in G$ we see that

$$d(R_h(f), R_h(f')) = \sup\{|f(xh) - f'(xh)| \mid x \in G\} = \sup\{|f(x) - f'(x)| \mid x \in G\} = d(f, f') < \varepsilon/2.$$

Hence

$$d(R_h(f'), R_g(f)) \leq d(R_h(f'), R_h(f)) + d(R_h(f), R_g(f)) < \varepsilon,$$

when $h \in U^{-1}g$. This proves continuity.

4. Suppose G is a compact topological groups and assume there exists unique **right-invariant** integral on G i.e. there exists a unique linear mapping $I \colon \mathrm{Map}(G, \mathbb{R}) \to \mathbb{R}$ such that
1) $I(1) = 1$,
2) $I(f) \geq 0$ if $f \colon G \to \mathbb{R}$ is such that $f(g) \geq 0$ for all $g \in G$ and
3) $I(R_h(f)) = I(f)$ for all $f \in \mathrm{Map}(G, \mathbb{R}), h \in G$.

Prove that I is also **left-invariant** i.e.

$$I(L_h(f)) = I(f)$$

for all $f \in \mathrm{Map}(G, \mathbb{R}), h \in G$. (Hint: uniqueness of I.)

Solution: Fix $g \in G$ and define $J \colon \mathrm{Map}(G, \mathbb{R}) \to \mathbb{R}$ to be a composition $I \circ L_g$, where $L_g \colon \mathrm{Map}(G, \mathbb{R}) \to \mathrm{Map}(G, \mathbb{R})$ is defined as in exercise 1. Since L_g is linear (exercise 1) and I is linear by assumption, J is linear.
Also we observe that

$$J(1) = I(L_g(1)) = I(1) = 1,$$
$$J(f) = I(L_g(f)) \geq 0,$$

whenever $f \geq 0$ and for every $h \in G$

$$J(R_h f) = I(L_h(R_g(f)) = I(R_g(L_h(f)) = I(L_h(f)) = J(f).$$

Here we used the fact that $L_h \circ R_g = R_g \circ L_h$, which is easy to prove:

$$L_h(R_g(f))(x) = R_g(f)(hx) = f(hxg) = L_h(f)(xg) = R_g(L_h(f))(x).$$

Hence J satisfies the same conditions as I. By uniqueness it follows that $J = I$. Hence

$$I(L_h(f)) = J(f) = I(f).$$

Since this works for every $h \in G$ we are done.

5. Suppose G is a finite group (with discrete topology). Prove the existence and uniqueness of Haar integral for G. Give a precise formula for the Haar integral of continuous $f: G \to \mathbb{R}$.

Solution: Suppose $I: \mathrm{Map}(G, \mathbb{R}) \to \mathbb{R}$ satisfies the conditions for Haar integral. For $g \in G$ let $\chi_g \in \mathrm{Map}(G, \mathbb{R})$ be the characteristic function of the singleton $\{g\}$ i.e.

$$\chi_g(x) = \begin{cases} 1, & x = g, \\ 0, & x \neq g. \end{cases}.$$

Notice that χ_g is continuous, since G has discrete topology.

Then $\chi_g = R_{g^{-1}}\chi_e$, so $I(\chi(g)) = I(\chi_e) = c \in \mathbb{R}$ for every $g \in G$. Since

$$1 = \sum_{g \in G} \chi_g,$$

linearity of Haar integral implies that

$$1 = I(1) = \sum_{g \in G} I(\chi_g) = cn,$$

where $n = |G|$ is the cardinality of G. Hence $c = \frac{1}{n}$. For an arbitrary $f \in \mathrm{Map}(G, \mathbb{R})$ we notice that

$$f = \sum_{g \in G} f(g)\chi_g,$$

so

$$I(f) = \sum_{g \in G} \frac{f(g)}{|G|}.$$

This both proves the uniqueness and provides with the precise formula for the Haar integral in this case.

It remains to show that if we define $I: \mathrm{Map}(G, \mathbb{R}) \to \mathbb{R}$ by the formula

$$I(f) = \sum_{g \in G} \frac{f(g)}{|G|}$$

then mapping I satisfies condition for Haar integral. Obviously I is linear and positive. Also

$$I(1) = \sum_{g \in G} \frac{1}{|G|} = 1,$$

$$I(R_h(f)) = \sum_{g \in G} \frac{f(gh)}{|G|} = \sum_{g \in G} \frac{f(g)}{|G|} = I(f)$$

for any $h \in G$.

6. Suppose G is a compact group and assume the existence and uniqueness of Haar integral on G. Let $f: G \to \mathbb{R}$ be continuous. Prove that

$$\int_G f(g)dg = \int_G f(g^{-1})dg.$$

(Hint: uniqueness of Haar integral.)

Solution: By uniqueness of Haar integral it is enough to prove that the mapping $I\colon \mathrm{Map}(G,\mathbb{R}) \to \mathbb{R}$ defined by

$$I(f) = \int_G f(g^{-1})dg$$

satisfies all conditions for Haar integral. This is an easy verification and left to the reader.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.