Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 4
Solutions

1. Suppose X is a topological space and G a topological group. A mapping $\Psi\colon X \times G \to X$ is called **right action** of G on X if the identities
1) $\Psi(x, e) = x$,
2) $\Psi(\Psi(x, g), g') = \Psi(x, gg')$
are satisfied for all $x \in X, g, g' \in G$. If one uses notation $\Psi(x, g) = xg$, these requirements can be written in the form
$$xe = x,$$
$$(xg)g' = x(gg').$$
Suppose $\Phi\colon G \times X \to X$ is a (left) action of G on X (as in definition 1.1). Prove that the mapping $\widehat{\Phi}\colon X \times G \to X$ defined by
$$\widehat{\Phi}(x, g) = \Phi(g^{-1}, x)$$
is a right action of G on X.
Prove that the correspondence $\Phi \mapsto \widehat{\Phi}$ is a bijection between the set of all (left) actions of G on X and the set of all right actions of G on X. What is its inverse?

Solution: Suppose $\Phi\colon G \times X \to X$ is a (left) action of G on X and the mapping $\widehat{\Phi}\colon X \times G \to X$ is defined by
$$\widehat{\Phi}(x, g) = \Phi(g^{-1}, x).$$
Let us check that $\widehat{\Phi}$ is a right action. For all $x \in X, g, g' \in G$ we have
$$\widehat{\Phi}(x, e) = \Phi(e^{-1}, x) = \Phi(e, x) = x,$$
$$\widehat{\Phi}(x, gg') = \Phi((gg')^{-1}, x) = \Phi(g'^{-1}g^{-1}, x) = \Phi(g'^{-1}, \Phi(g^{-1}, x)) =$$
$$= \widehat{\Phi}(\Phi(g^{-1}, x), g') = \widehat{\Phi}(\widehat{\Phi}(x, g), g').$$
Hence $\widehat{\Phi}$ is a right action.

Similarly if $\Psi\colon X \times G \to X$ is a right action we can define $\widetilde{\Psi}\colon G \times X \to X$ by the formula $\widetilde{\Psi}(g, x) = \Psi(x, g^{-1})$. As above it is easy to check that $\widetilde{\Psi}$ is a left action, if Ψ is a right action. Also since $(g^{-1})^{-1} = g$ for all $g \in G$, it is easy to verify that the correspondences $\Phi \mapsto \widehat{\Phi}$ and $\Psi \mapsto \widetilde{\Psi}$ are inverses of each other.

2. Suppose G is a topological group and H is its subgroup. Prove that the mapping $\Phi\colon H \times G \to G$, $\Phi(h, g) = hg$ is a (left) action of H on G and the mapping $\Psi\colon G \times H \to G$, $\Psi(g, h) = gh$ is a right action of H on G.

Suppose $g \in G$. What is the isotropy subgroup H_g with respect to action Φ?

What is the orbit space defined by this action?

Can you come up with the example of a (left) action of H on G such that the orbit space induced by this action would be precisely coset space G/H?

Solution: For all $g \in G, h, h' \in H$ we have

$$eg = g,$$
$$(hh')g = h(h'g),$$

so Φ is a left action. Similarly we see that Ψ is a right action.

Suppose $g \in G$. Then $h \in H_g$ if and only if $hg = g$, which implies (multiply by g^{-1} from the right in the group G) that $h = e$. Hence the isotropy group H_g is a trivial group $\{e\}$ for all $g \in G$.
The orbit of an element $g \in G$ is by definition the set

$$Hg = \{hg \mid h \in H\}$$

i.e. the **the right** coset of g with respect to H. Hence the orbit space is the space of right cosets $H \setminus G$ (with topology naturally defined by the canonical projection $\pi\colon G \to H \setminus G$.

Similarly it is easy to realise that the orbit of $g \in G$ with respect to right action Ψ is a left coset

$$gH = \{gh \mid h \in H\}.$$

Corresponding left and right actions (see exercise 1) clearly have the same orbit space, so we see immediately that the left action $\widetilde{\Psi}$ of H on G defined by $\widetilde{\Psi}(h, g) = \Psi(g, h^{-1}) = gh^{-1}$ is a left action, which orbit space is the space of left cosets G/H.

3. Suppose G is a topological group and H is its subgroup. Consider the canonical action of G on the coset space G/H defined by $g \cdot g'H = (gg')H$ (example III.1.3).

   Let $x = gH \in G/H$ be arbitrary. What is the isotropy subgroup G_x?
   Prove that the kernel of this action is the biggest normal subgroup of G contained in H (i.e. the kernel K is a normal subgroup of G, $K \subset H$ and if L is a normal subgroup of G such that $L \subset H$, then $L \subset K$).

   Solution: Let $x = gH \in G/H$, $g \in G$. Then $h \in G_x$ if and only if

   $$h(gH) = gH,$$

   i.e. if and only if $g^{-1}hg \in H$ if and only if $g \in gHg^{-1}$. Hence $G_x = gHg^{-1}$.

   The kernel of the action is then the intersection of all isotropy groups i.e. the subgroup

   $$K = \cap_{g \in G} gHg^{-1}.$$

   K is normal, since it is a kernel of (algebraic) homomorphism $G \to \mathrm{Homeo}(X)$ defined by $g \mapsto \Phi_g$. Since $H = eHe^{-1}$ is one of the sets in the intersection

$\cap_{g\in G} gHg^{-1}$, it follows that $K \subset H$. Suppose L is normal in G and $L \subset H$. Since L is normal for every $g \in G$

$$L = gLg^{-1} \subset gHg^{-1},$$

so $L \subset K$.

4. Consider the action of the special linear group $G = SL(n;\mathbb{R})$ on $X = \mathbb{R}^n$ defined as usual by $A \cdot x = Ax$, $A \in SL(n;\mathbb{R}), x \in \mathbb{R}^n$.
What are the orbits of this action? What is the orbit space X/G? Is it Hausdorff? T_1? T_0?.
Is canonical projection $X \to X/G$ a closed mapping?

Solution: Consider first the case $n = 1$. Then $SL(n;\mathbb{R}) = [1]$ is a trivial group, so orbits are singletons $\{x\}$, $x \in \mathbb{R}$ and the orbit space is trivially $\mathbb{R}$ (no non trivial identifications). This space is Hausdorff, T_1 and T_0. The canonical projection $\mathbb{R} \to \mathbb{R}$ is just identity mapping, so is closed.

Suppose now $n \geq 2$. We claim that there exists precisely two orbits. Clearly $A0 = 0$ for all $A \in SL(n;\mathbb{R})$ so the orbit $G0$ of the origin $0 \in \mathbb{R}^n$ is just s singleton $\{0\}$. It remains to show that all other points are in the same orbit. It is enough to show that every $x \in \mathbb{R}^n \setminus \{0\}$ is in the orbit of $e_1 = (1, 0, \ldots, 0)$. Suppose $x \in \mathbb{R}^n$, $x \neq 0$. Then there exists a linear basis $x_1, \ldots, x_n$ of R^n such that $x = x_1$. Let A be an $n \times n$-matrix whose ith column is precisely x_i. Then $A(e_1) = x_1$ by construction and A is invertible (since its column are linearly independent) i.e. $\det A \neq 0$. This matrix is not necessarily an element of $SL(n;\mathbb{R})$, but if we multiply the last n-th column by $\frac{1}{\det A}$ we obtain a matrix B such that $\det B = \frac{\det A}{\det A} = 1$, since det is multilinear function of the columns. Hence $B \in SL(n;\mathbb{R})$. Since $n \geq 2$ the first column of B is the same as the first column of A, i.e. x. Hence $B(e_1) = x$, so every $x \neq 0$ is in the same orbit as e_1.

It follows that X/G is s two point space $Y = \{a, b\}$, where $a = \mathbb{R}^n \setminus \{0\}, b = \{0\}$. By checking the inverse images of subsets of Y in X we see that the only open subsets of Y are $\emptyset, Y$ and the singleton $\{a\}$. Consequently Y is not Hausdorff or even T_1 (singleton $\{a\}$ is not closed). It is T_0, since a has a neighbourhood $\{a\}$ that does not include b. The projection mapping $p\colon X \to X/G$ is not closed, since for every $x \neq 0$ the image of the closed set $\{x\}$ is $\{a\}$, which is not closed.

5. Consider the action of the orthogonal linear group $G = O(n)$ on $X = \mathbb{R}^n$ defined as usual by $A \cdot x = Ax$, $A \in O(n), x \in \mathbb{R}^n$.
What are the orbits of this action? What is the orbit space X/G? Is it Hausdorff? T_1? T_0?.

$O(n)$ also acts on S^{n-1} by the same formula. What is the orbit space of this action?

Solution: Suppose x and y are in the same orbit. Then there exists ortogonal $A \in O(n)$ such that $Ax = y$. Since A preserves norms (exercise 2.2)

$$|x| = |Ax| = |y|.$$

Let us prove the opposite. Suppose $|x| = |y| = r \geq 0$. We claim that x and y are in the same orbit. If $r = 0$, then $x = y$ and the claim is clear. Suppose $r > 0$. It is enough to show that any point x with $|x| = r$ is in the orbit of re_1. Let $x' = x/r$, then $|x'| = 1$, so by linear algebra there exists an orthonormal basis $a_1, \ldots, a_n$ of $\mathbb{R}^n$ such that $x' = a_1$. Let A be an $n \times n$ matrix, whose ith column is a_i for all $i = 1, \ldots, n$. Then $A \in O(n)$ by construction (exericise 2.2.) and $A(e_1) = a_1$. Since A acts linearly $A(re_1) = rA(e_1) = ra_1 = x$. The claim is proved.

Hence the orbits of this action are circles

$$S_r = \{x \in \mathbb{R}^n \mid |x| = r\}, r \geq 0$$

centered at origin (including degenerated case $r = 0$). Define a mapping $f\colon X \to \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ by $f(x) = |x|$. Then f is continuous surjective and $f(x) = f(y)$ if and only if x and y are in the same orbit of the action, hence f induces continuous bijection $\tilde{f}\colon X/G \to \mathbb{R}_+$ such that $\tilde{f} \circ \pi = f$. Here $\pi\colon X \to X/G$ is a canonical projection.
To show that $\tilde{f}$ is actually homeomorphism, we notice that f has a left inverse $j\colon \mathbb{R}_+ \to X$ defined by $j(r) = (r, 0, \ldots, 0)$. Then $f \circ j = \text{id}$, so $\tilde{f} \circ (\pi \circ j) = id$. Since $\tilde{f}$ is bijection it follows that $pi \circ j$ is its inverse. Since this inverse is continuous, it follows that $\tilde{f}$ is homeomorphism.

Hence X/G is homeomorphic to $\mathbb{R}_+$, so is Hausdorff, T_1 and T_0.
Alternatively one can use Theorem 1.11(1), because $O(n)$ is compact.

When we consider the restricted action of $O(n)$ on S^{n-1} it follows from the observations above that there is only one orbit. Hence the orbit space in this case is a singleton space.

6.

6. Consider the action of the orthogonal linear group $G = O(n)$ on $X = \mathbb{R}^n$ defined as above by $A \cdot x = Ax$, $A \in O(n), x \in \mathbb{R}^n$.

a) Prove that the isotropy group G_{e_n} is isomorphic (as a topological group) to $O(n-1)$.
Here $e_n = (0, \ldots, 0, 1)$.

b) Suppose $x \in \mathbb{R}^n, x \neq 0$. Prove that the isotropy group G_x is isomorphic (as a topological group) to $O(n-1)$. (Hint: a) and Lemma 1.17). What about the isotropy group G_0?

Solution: a) Suppose an orthogonal matrix $A \in O(n)$ is in G_{e_n} i.e. $Ae_n = e_n$. This means precisely that the last column of A is $e_n = (0, \ldots, 0, 1)$. Since A is orthogonal the set $A(e_1) = a_1, A(e_2) = a_2, \ldots, A(e_i) = a_i, \ldots, A(e_n) = e_n$ of the columns of A (regarded as vectors of $\mathbb{R}^n$ is orthonormal (exercise 2.2), so in particular $a_i \cdot e_n = 0$ for $i < n$, so it follows that the last row of A is

$(0,\dots,1)$, i.e. A looks like a matrix

$$A = \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix},$$

where A' is $(n-1)\times(n-1)$ matrix. It is easy to see that A' is orthonormal. Conversely any matrix of the type

$$A = \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix},$$

where $A' \in O(n-1)$ is orthonormal and $Ae_n = e_n$. It follows that the mapping $\alpha\colon O(n-1) \to G_{e_n}$ defined by

$$A' \mapsto \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix}$$

is an isomorphism of groups. It is easy to check that it is homeomorphism as well. Hence G_{e_n} is isomorphic to $O(n-1)$ as a topological group.

Suppose $x \neq 0$ and let $k = |x| > 0$. By exercise 5 above x and ke_n are in the same orbit. By Lemma 1.17 G_x and G_{ke_n} are conjugate to each other, hence isomorphic as topological group (since conjugation in a topological group is clearly homeomorphism and isomorphism of groups). It remains to prove that $G_{ke_n} = G_{e_n} \cong O(n-1)$. Suppose $A \in O(n)$, $A(ke_n) = ke_n$. Then by linearity of A we have $kA(e_n) = ke_n$, which implies ($k \neq 0$!), that $A(e_n) = e_n$. Hence $A \in G_{e_n}$. Conversely if $Ae_n = e_n$, then by linearity $A(ke_n) = kA(e_n) = ke_n$. We are done.

Since 0 is fixed by any matrix $A \in O(n)$, it follows that $G_0 = O(n)$ is the whole group.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.