Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 4
13-17.02.2012

1. Suppose X is a topological space and G a topological group. A mapping $\Psi\colon X \times G \to X$ is called **right action** of G on X if the identities
1) $\Psi(x, e) = x$,
2) $\Psi(\Psi(x, g), g') = \Psi(x, gg')$
are satisfied for all $x \in X, g, g' \in G$. If one uses notation $\Psi(x, g) = xg$, these requirements can be written in the form

$$xe = x,$$

$$(xg)g' = x(gg').$$

Suppose $\Phi\colon G \times X \to X$ is a (left) action of G on X (as in definition 1.1). Prove that the mapping $\widehat{\Phi}\colon X \times G \to X$ defined by

$$\widehat{\Phi}(x, g) = \Phi(g^{-1}, x)$$

is a right action of G on X.
Prove that the correspondence $\Phi \mapsto \widehat{\Phi}$ is a bijection between the set of all (left) actions of G on X and the set of all right actions of G on X. What is its inverse?

2. Suppose G is a topological group and H is its subgroup. Prove that the mapping $\Phi\colon H \times G \to G$, $\Phi(h, g) = hg$ is a (left) action of H on G and the mapping $\Psi\colon G \times H \to G$, $\Psi(g, h) = gh$ is a right action of H on G.

Suppose $g \in G$. What is the isotropy subgroup H_g with respect to action Φ?
What is the orbit space defined by this action?

Can you come up with the example of a (left) action of H on G such that the orbit space induced by this action would be precisely coset space G/H?

3. Suppose G is a topological group and H is its subgroup. Consider the canonical action of G on the coset space G/H defined by $g \cdot g'H = (gg')H$ (example III.1.3).

Let $x = gH \in G/H$ be arbitrary. What is the isotropy subgroup G_x?
Prove that the kernel of this action is the biggest normal subgroup of G contained in H (I.e. the kernel K is a normal subgroup of G, $K \subset H$ and if L is a normal subgroup of G such that $L \subset H$, then $L \subset K$).

4. Consider the action of the special linear group $G = SL(n; \mathbb{R})$ on $X = \mathbb{R}^n$ defined as usual by $A \cdot x = Ax$, $A \in SL(n; \mathbb{R}), x \in \mathbb{R}^n$.

What are the orbits of this action? What is the orbit space X/G? Is it Hausdorff? T_1? T_0?.
Is canonical projection $X \to X/G$ a closed mapping?

5. Consider the action of the orthogonal linear group $G = O(n)$ on $X = \mathbb{R}^n$ defined as usual by $A \cdot x = Ax$, $A \in O(n), x \in \mathbb{R}^n$.
What are the orbits of this action? What is the orbit space X/G? Is it Hausdorff? T_1? T_0?.

   $O(n)$ also acts on S^{n-1} by the same formula. What is the orbit space of this action?

6. Consider the action of the orthogonal linear group $G = O(n)$ on $X = \mathbb{R}^n$ defined as above by $A \cdot x = Ax$, $A \in O(n), x \in \mathbb{R}^n$.

   a) Prove that the isotropy group G_{e_n} is isomorphic (as a topological group) to $O(n-1)$.
Here $e_n = (0, \ldots, 0, 1)$.

   b) Suppose $x \in \mathbb{R}^n, x \neq 0$. Prove that the isotropy group G_x is isomorphic (as a topological group) to $O(n-1)$. (Hint: a) and Lemma 1.17). What about the isotropy group G_0?

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.