Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 3
06.02-10.02.2012

1. a) A subset A of a topological space X is called **locally closed** if every point x of A has an open neighbourhood U (in X) such that $U \cap A$ is closed in U. Prove that $A \subset X$ is locally closed if and only if A is open in its closure A.

   b) Suppose G is a topological group and H its subgroup, which is locally closed in G. Prove that H is a closed subgroup of G. (Hint: use the fact that open subgroup is closed).

2. a) Suppose A is a locally compact subspace of a Hausdorff space X. Prove that A is locally closed in X.

   b) Suppose H is a locally compact subgroup of a topological group G. Prove that H is closed in G. Conclude that every discrete subgroup of a topological group is closed.

3. Suppose G is a compact group and $\phi\colon \mathbb{Z} \to G$ is an injective homomorphism. Prove that ϕ **is not** embedding i.e. homeomorphism to its image $\phi(\mathbb{Z})$. (Hint: otherwise $\phi(\mathbb{Z})$ is a closed discrete subgroup of G.)
   Construct a concrete example of an injective homomorphism $\phi\colon \mathbb{Z} \to G$, where G is compact group.

4. Suppose G, G' are topological groups and $f\colon G \to G'$ is a homomorphism of groups which is continuous as a mapping between topological spaces. Prove that
   $$N = \mathrm{Ker}(f) = \{g \in G \mid f(g) = e'\}$$
   (where e' is a neutral element of G') is a closed normal subgroup of G. Prove that the induced homomorphism
   $$\tilde{f}\colon G/N \to G'$$
   (defined by $\tilde{f}(gN) = f(g)$) is a continuous injective mapping. Is it necessarily a homeomorphism to its image $f(G')$?

5. Suppose G, G' are topological groups and $f\colon G \to G'$ is a surjective continuous homomorphism. Prove that f is a quotient mapping if and only if f is an open mapping.

6. Suppose G is a connected topological group and N its discrete normal subgroup. Prove that N is **central** in G i.e.
   $$xy = yx \text{ for all } x \in G, n \in N.$$
   (Hint: consider the mapping $f\colon G \to N$, $f(g) = gng^{-1}$, where $n \in N$.)

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.