Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 1
Solutions

1. Prove that $\mathbb{R}^n$, $n \in \mathbb{N}$ equipped with standard topology and addition of vectors
$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
is a topological group.

Solution: $\mathbb{R}^n$ is Hausdorff, in particular T_1.
Addition mapping $\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by
$$\mu((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
is clearly continuous, since its components are combinations of projections and addition of real numbers, which is known to be continuous.
Inverse element mapping $\iota\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by
$$\iota((a_1, a_2, \ldots, a_n) = (-a_1, -a_2, \ldots, -a_n).$$
Again the components of this mapping are compositions of projection and inverse element-mapping for the group $(\mathbb{R}, +)$, which is known to be continuous.

2. Consider $\mathbb{R}^2$ as the set of complex numbers $\mathbb{C}$, equipped with the standard multiplication of complex numbers defined by
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$
Show that $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a topological group, if equipped with this multiplication and standard topology as a subset of $\mathbb{R}^2$. Why did we have to exclude 0?

Solution: $\mathbb{R}^2$ is Hausdorff, in particular T_1, so its subset $\mathbb{R}^2 \setminus \{0\}$ is also T_1.
Multiplication of complex numbers is clearly a continuous mapping $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, since its components are combinations of projection mappings, multiplication, addition and subtraction of real numbers, which are known to be continuous.
It remains to probe that the inverse $\iota\colon \mathbb{C}^* \to \mathbb{C}^*$ is a continuous operation. From algebra it is known that an inverse element of the complex number $(a, b) \neq 0$ is the complex number $(b/(a^2 + b^2), -a/(a^2 + b^2))$. Hence
$$\iota(a, b) = (b/(a^2 + b^2), -a/(a^2 + b^2)).$$
This clearly is a continuous mapping $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$.

3. Topological space X is called **homogeneous** if for every pair of points $x, y \in X$ there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$. Prove that every topological group is homogeneous as a topological space.
Conclude that the unit interval $I = [0, 1]$ (equipped with its standard topology) cannot be given a structure of a topological group.

Solution: Suppose $x, y \in G$, G a topological group. Left translation $l_{yx^{-1}}: G \to G$ defined by $l_{yx^{-1}}(g) = yx^{-1}g$ is a homeomorphism (Lemma 1.2) and

$$l_{yx^{-1}}(x) = y.$$

Hence G is homogeneous.

The unit interval I is not homogeneous space, since there is no homeomorphism $f: I \to I$ such that $f(0) = f(1/2)$. Let us prove this via counter assumption. Suppose f is such a mapping. Then f defines by restriction a homeomorphism between spaces $I \setminus \{0\}$ and $I \setminus \{1/2\}$. But $I \setminus \{0\} =]0, 1]$ is an interval, hence connected space, while $I \setminus \{1/2\} = [0, 1/2[\cup]1/2, 1]$ is not connected.

4. a) Suppose G is a topological group, U an open neighbourhood of the neutral element $e \in G$ and W an open neighbourhood of e such that

$$WW^{-1} \subset U.$$

Prove that $\overline{W} \subset U$. (Hint: xW is a neighbourhood of any point $x \in \overline{W}$).
b) Topological space X is called **regular** if for any $x \in X$ and any open neighbourhood U of x there exists a neighbourhood W of x such that $\overline{W} \subset U$. Prove that every topological group is regular as a topological space.

Solution: Suppose $x \in \overline{W}$. Since W is a neighbourhood of e, xW is a neighbourhood of x in G. From the definition of the closure it follows that $xW \cap W \neq \emptyset$. Hence there exist $w_1, w_2 \in W$ such that $xw_1 = w_2$. This implies that

$$x = w_2 w_1^{-1} \in WW^{-1} \subset U.$$

Hence $x \in U$.

b) Suppose G is a topological group and $x \in G$. Suppose U is a neighbourhood of x. Then $x^{-1}U$ is a neughbourhood of the neutral element e. By Lemma 1.7 there exists a neighbourhood W of e such that $WW^{-1} \subset x^{-1}U$. By a)

$$\overline{W} \subset x^{-1}U.$$

Now $xW = l_x(W)$ is a neighbourhood of x in G. The left translation l_x is a homeomorphism, hence

$$\overline{xW} = \overline{l_x(W)} = l_x(\overline{W}) = x\overline{W} \subset xx^{-1}U = U.$$

5. Suppose X is a T_1 and regular topological space. Prove that X is Hausdorff. Conclude that every topological group is Hausdorff as a topological space.

Solution: Suppose X is T_1 and regular. Suppose $x, y \in X$. Since X is T_1 the singleton $\{y\}$ is closed in X, hence $U = X \setminus \{y\}$ is an open neighbourhood of x in X. Since X is regular, there exist an open neighbourhood W of x such that $\overline{W} \subset U$. It follows that W and $X \setminus \overline{W}$ are disjoint open neighbourhoods of x and y. Hence X is Hausdroff.

Every topological group G is T_1 by the definition. Also the previous exercise shows that G is regular topological space. Hence G is Hausdorff.

6. A topological space is said to be T_0-space, if for every pair of points x, y there exists a neighbourhood U of one of the points, that do not contain the other point.
a) Suppose X is T_0 and regular. Prove that X is Hausdorff.
b) Suppose $(G, \cdot, \tau)$ is a triple, where $(G, \cdot)$ is a group and τ is a topology in the set G. Denote by $f\colon G \times G \to G$ the mapping defined by
$$f(x, y) = xy^{-1}.$$
Prove that G is a topological group if and only if G is T_0 as a topological space and f is continuous.

Solution: a) Prove is similar to the proof in the solution of the previous exercise. Suppose X is T_0 and regular. Suppose $x, y \in X$. Since X is T_0 one of the points x or y has a neighbourhood U which does not contain the other point. By the symmetry we may assume $x \in U$. Since X is regular, there exist an open neighbourhood W of x such that $\overline{W} \subset U$. It follows that W and $X \backslash \overline{W}$ are disjoint open neighbourhoods of x and y. Hence X is Hausdroff.

b) Suppose G is a topological group. Then G is T_1, hence in particular T_0 (Topology II). Also f can be written as the composition $\mu \circ g$, where $\mu\colon G \to G$ is a multiplication, $\mu(x, y) = xy$ and $g\colon G \times G \to G \times G$ is defined by $g(x, y) = (x, y^{-1}) = (x, \iota(y))$, where $\iota\colon G \to G$ is inverse element mapping, $\iota(x) = x^{-1}$. Now μ is continuous by assumption, also ι is, so it is easy to see that g is also continuous (its projections are). Hence f is continuous.

Conversely suppose f is continuous and G is T_0. Let us prove first that algebraic operations $\mu\colon G \times G \to G$, $\mu(x, y) = xy$ and $\iota\colon G \to G$, $\iota(x) = x^{-1}$ are continuous. Now ι can be written as the composition $f\colon h$, where $h\colon G \to G \times G$, $h(x) = (e, x)$. The mapping h is clearly continuous (here e is the neutral element of the group G), hence ι is continuous.
Now we see that μ can be written as the composition of the continuous mappings $k\colon G \times G \to G \times G$ and f, where
$$k(x, y) = (x, y^{-1}) = (x, \iota(y)).$$
Since we have already shown that ι is continuous, k is continuous, hence also μ is.

It remains to show that G is T_1. Since G is T_0 by a) it is enough to show it is regular. We have to go through the prove of regularity of every topological group (see exercise 4) and verify that it does not use T_1-assumption. First of all mapping f is continuous, so for every neighbourhood U of identity element $e \in G$ there exists a neighbourhood W of e, so that
$$WW^{-1} = f(W \times W) \subset U.$$
In other words Lemma 1.7 is still true. Next we notice that $\overline{W} \subset U$ - the proof is the same as in exercise 4. Finally since multiplication is continuous all left translations l_x are continuous. Moreover every translation is bijection and translation $l_{x^{-1}}$ is the inverse of translation l_x. It follows that every translation l_x is a homeomorphism. Hence for every x the set xW is a neighbourhood of

x and

$$\overline{xW} = x\overline{W} \subset xU.$$

Now for arbitrary neighbourhood V of x $U = x^{-1}W$ is a neighbourhood of e, hence $xU = V$ and we are done.