Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 13
Solutions

1. Suppose X is a G-space and H is a closed subgroup of G. A subset S of X is called *an H-kernel* if there exists a G-mapping $f\colon X \to G/H$ such that $S = f^{-1}(eH)$. Suppose S is an H-kernel in X. Prove the following:
   i) S is closed in X.
   ii) S is H-equivariant.
   iii) $(S|S) = \{g \in G \mid gS \cap S \neq \emptyset\} = H$.
   iv) A G-mapping $f'\colon X \to G/H$ such that $S = f'^{-1}(eH)$ is unique.

   Solution: i) Since H is closed, G/H is T_1, so $\{eH\}$ is closed in G/H. Thus $S = f^{-1}(eH)$ is closed.
   ii) Suppose $h \in H$, $s \in S$. Then

   $$f(hs) = hf(s) = heH = H,$$

   so $hs\ inf^{-1}(eH) = S$.
   iii) By ii) $H \subset G(S|S)$. Conversely suppose $g \in G(S|S)$, i.e. $gs_1 = s_2$ for some $s_1, s_2 \in S$. Then

   $$gH = gf(s_1) = f(gs_1) = f(s_2) = H,$$

   so $g \in H$.
   iv) Suppose f and f' both G-mappings $X \to G/H$ such that $S = f'^{-1}(eH)$. Suppose $x \in X$ and $f(x) = gH$, $g \in G$. Then $f(g^{-1}x) = H$, so $g^{-}x \in S$, hence

   $$f'(x) = gf'(g^{-1}x) = gH = gf(g^{-1}x) = f(x).$$

2. Suppose X is a G-space, where G is compact, H a closed subgroup of G and $S \subset X$ is an H-kernel. Prove that the mapping $\alpha\colon G \underset{H}{\times} S \to X$, $\alpha[g, s] = gs$ is a G-homeomorphism.

   Solution: Since G is compact the action mapping $\Phi\colon G \times X \to X$ is closed (Topological Transformation Groups I, Theorem 1.9). Since S is closed by the previous exercise, restriction $\Psi = \Phi|G \times S\colon G \times S \to X$ is also closed mapping.
   Next we show that Ψ is surjection. Suppose $x \in X$ and let $g \in G$ be such that $f(x) = gH$. Then $f(g^{-1}x) = H$, i.e. $g^{-1}x \in S$. Hence $x = g(g^{-1}x) = \Psi(g, g^{-1}x)$. Hence Ψ is a surjection. Since it is also closed, we conlude that Ψ is a quotient mapping.
   Suppose $\Psi(g, s) = gs = g's' = \Psi(g', s')$. Then $(g'^{-1}g)s = s'$ i.e. $g'^{-1}g = G(S|S)$, which equals H by the previous exercise. Hence $g'^{-1}g = h \in H$, so

   $$(g', s') = (gh^{-1}, hs).$$

Conversely $\Psi(gh^{-1}, hs) = \Psi(g,s)$ for all $g \in G, h \in H, s \in S$. Thus we see that the quotient space mapping Ψ defines is precisely the induced space $G \underset{H}{\times} S$, hence the induced mapping is exactly $\alpha\colon G \underset{H}{\times} S \to X$, which is thus a homeomorphism.

3. Suppose X is a G-space, H a closed subgroup of G and $S \subset X$ is an H-kernel. Denote by $f\colon G \to G/H$ a G-mapping such that $f^{-1}(eH) = S$.
Suppose canonical projection $\pi\colon G \to G/H$ admits a local cross-section $c\colon U \to G$, $U \subset G/H$. Define $\alpha\colon G \underset{H}{\times} S \to X$, $\alpha[g,s] = gs$ as above and let $\beta\colon f^{-1}(U) \to G \underset{H}{\times} S$ be defined by

$$\beta(x) = [c(f(x)), (cf(x))^{-1}x].$$

Show that β is an inverse of the restriction of α on the subset

$$\{[g,s] \mid \ gH \in U\} \subset G \underset{H}{\times} S.$$

Conclude that α is a G-homeomorphism.

Solution: α is a mapping induced by an inclusion $S \hookrightarrow X$ by the universal property of induced space, so it is well-defined and continuous G-mapping.
β is well-defined, since $f((cf(x))^{-1}x) = (cf(x))^{-1}f(x) \in H$ by the basic properties of local cross-sections (see exercise 12.1), so $(cf(x))^{-1}x \in S$. Continuity of S is clear. Also we notice that β maps $f^{-1}(U)$ to the set $\{[g,s] \mid \ gH \in U\} \subset G \underset{H}{\times} S$, because if $f(x) \in U$, then $cf(x)H = \pi(c(f(x)) = f(x)$.
Conversely if $gH \in U$, then $f(\alpha([g,s])) = f(gs) = gH \in U$, i.e. α maps $\{[g,s] \mid \ gH \in U\}$ to $f^{-1}U$.
It remains to show that β and $\alpha|\{[g,s] \mid \ gH \in U\}$ are inverses of each other.
Suppose $[g,s]$ is such that $gH \in U$. Then

$$\beta(\alpha([g,s]))) = \beta(gs) = [c(f(gs)), (cf(gs))^{-1}gs] = [c(g(f(s)), (c(g(f(s))^{-1}gs] = [c(gH), (cgH)^{-1}gs].$$

By the exercise 12.1 there exists $h \in H$ such that $gh^{-1} = c(gH)$, hence

$$[c(gH), (cgH)^{-1}gs] = [gh^{-1}, hg^{-1}gs] = [gh^{-1}, hs] = [g,s].$$

Suppose $x \in f^{-1}U$. Then

$$\alpha(\beta(x)) = \alpha[c(f(x)), (cf(x))^{-1}x] = c(f(x))(cf(x))^{-1}x = x.$$

4. Suppose X is a G-space and $S \subset X$ is an G_x-kernel for some $x \in S$. Prove that $[G_x] \geq [G_y]$ for every $y \in X$.

Solution: Suppose $y \in X$ and let $g \in G$ be such that $f(y) = gG_x$, where $f\colon X \to G/G_x$ is a G-equivariant mapping such that $S = f^{-1}(G_x)$ Since f is G-equivariant,

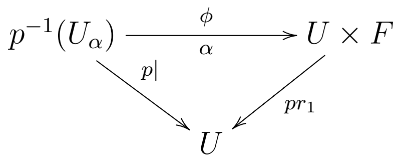
$$G_y \subset G_{f(y)} = gG_xg^{-1}.$$

Hence G_y is conjugate to a subgroup of G_x, i.e. $[G_x] \geq [G_y]$.

5. Suppose X is a G-space, H a closed subgroup of G and $S \subset X$ is an H-kernel. Suppose canonical projection $\pi\colon G \to G/H$ admits a local cross-section $s\colon U \to G$, $U \subset G/H$.
Prove that inclusion $i\colon S \hookrightarrow X$ induces a homeomorphism $S/H \to X/G$.

Solution: By Proposition 2.9 in "Induced space and twisted products"lecture material $S/H \cong (G \underset{H}{\times} S)/G$. By the exercise 3 $G \underset{H}{\times} S \cong X$ as G-spaces, so $G \underset{H}{\times} S)/G \cong X/G$. Combining these two together yields the result.

6. Suppose X, E, F are topological spaces and $p\colon E \to X$ is a continuous mapping. We say that p is *locally trivial with fibre* F if there exists an open covering $(U_\alpha)_{\alpha\in\mathcal{A}}$ of X and a homeomorphism $\phi_\alpha\colon p^{-1}U_\alpha \to U_\alpha \times F$ for every $\alpha \in \mathcal{A}$ such that the diagram



commutes.
a) Prove that p is an open mapping.
b) Suppose $\alpha, \beta \in \mathcal{A}$ and $x \in U_\alpha \cap U_\beta$. Prove that the *transition function* $\theta^x_{\beta\alpha}\colon F \to F$ defined by

$$\theta^x_{\beta\alpha}(f) = pr_2(\phi_\beta(\phi_\alpha^{-1}(x, f)))$$

is a well-defined homeomorphism.
c) Suppose $\alpha, \beta \in \mathcal{A}$ and $U_\alpha \cap U_\beta \neq \emptyset$. Prove that

$$\phi_\beta \circ \phi_\alpha^{-1}(x, f) = (x, \theta^x_{\beta\alpha}(f))$$

for all $x \in U_\alpha \cap U_\beta, f \in F$.

Solution: a) Since $p^{-1}(U_\alpha)$ consitute open covering of E, it is enough to show that $p|p^{-1}(U_\alpha) \to U_\alpha$ is open. But this mapping is a composition of a homeomorphism ϕ_α and a projection pr_1, which is an open mapping.

b) Suppose $y = \phi_\alpha^{-1}(x, f) \in p^{-1}(U)$. Since $x = pr_1(\phi_\alpha(y)) = p(y)$, it follows that $y \in p^{-1}(x) \subset p^{-1}(U_\beta)$. Hence $\phi_\beta(y) \in U_\beta \times F$ exists and $pr_2(\phi_\beta(y)) \in F$. Hence $\theta^x_{\beta\alpha}(f)$ is well-defined. It is obviously continuous. Moreover if we interchange α and β we see that $\theta^x_{\alpha\beta}$ exists and is a continuous inverse for $\theta^x_{\beta\alpha}$.

c) By b) it is enough to show that $pr_1(\phi_\beta \circ \phi_\alpha^{-1}(x, f)) = x$. But

$$pr_1(\phi_\beta \circ \phi_\alpha^{-1}(x, f)) = p(\phi_\alpha^{-1}(x, f)) = pr_1(x, f) = x.$$

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.