Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 12
Solutions

1. Suppose G is a topological group and H closed subgroup of G. A local cross-section of canonical projection $\pi\colon G \to G/H$ is a continuous mapping $s\colon U \to G$ defined on some open non-empty subset U of G/H such that $\pi(s(x)) = x$ for all $x \in U$.

   If there exists at least one local cross-section of $\pi\colon G \to G/H$ we say that canonical projection π admits local cross-section.

   a) Suppose $\pi\colon G \to G/H$ admits local cross-section. Prove that every point $x \in G/H$ has an open neighbourhood U such that there exists local cross section $s\colon U \to G$ of π.

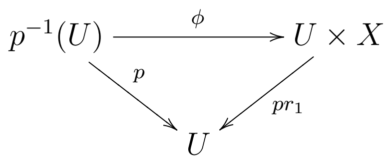
   b) Suppose $s\colon U \to G$ is a local cross-section of π, where U is open subset of G/H. Suppose $g \in \pi^{-1}(U)$ and let $g' = s(\pi(g))$. Prove that $gH = g'H$.

   c) Suppose $\pi\colon G \to G/H$ admits local cross-section. Let $g \in G$. Prove that there exists a local cross section $s\colon U \to G$ of π such that $g \in \pi^{-1}U$ and $s(gH) = g$.

   Solution: a) Suppose V is a non-empty open subset of G/H such that there exists a local cross-section $t\colon V \to G$. Suppose $x \in G/H$. Then there exists $g \in G$ such that $gx \in V$ (since $V \neq \emptyset$ and G/H has only one G-orbit). The set $g^{-1}V = U$ is then a neighbourhood of x and we can define a mapping $s\colon U \to G$ by the formula
$$s(y) = g^{-1}s(gy).$$
   Mapping s is evidently continuous and
$$\pi(s(y)) = \pi(g^{-1}s(gy)) = g^{-1}\pi(s(gy)) = g^{-1}(gy) = y$$
   for all $y \in U$.

   b) Since s is a cross-section
$$g'H = \pi(g') = \pi s(\pi(g)) = \pi(g) = gH.$$
   c) By a) there exists a local cross-section $s'\colon U \to G$ defined on a neighbourhood U of $\pi(g) = gH$. Let $g' = s'(gH)$, then by b) $g'H = gH$, i.e. $g'^{-1}g \in H$. Now define $s\colon U \to G$ by the formula
$$s(y) = s'(y)g'^{-1}g.$$
   Then s is continuous, $s(gH) = s'(gH)g'^{-1}g = g'(g'^{-1}g) = g$ and
$$\pi(s(y)) = s'(y)(g'^{-1}g)H = s'(y)H = \pi(s(y)) = y.$$

2. Suppose G is a topological group, H is its closed subgroups and suppose projection $\pi\colon G \to G/H$ admits local cross-section. Suppose X is an H-space and let $p\colon G \underset{H}{\times} X \to G/H$, $p([g,x]) = gH$.
Let $s\colon U \to G$ be a local cross section of π.
Define $\phi\colon p^{-1}U \to U \times X$, $\phi([g,x]) = (gH, (s(gH)^{-1}g)x)$, $\psi\colon U \times X \to p^{-1}U$, $\psi(gH,x) = [s(gH),x]$. Prove that ϕ and ψ are well-defined continuous G-mappings and inverses of each other (hence G-homeomorphisms) and the diagram



commutes.

Solution: First we check that ϕ is well-defined. First of all by the exercise 1b) $(s(gH)^{-1}g) \in H$ for all $g \in \pi^{-1}U$, so $(s(gH)^{-1}g)x$ makes sense for all $x \in X$ (which is only H-space!).

Let $q\colon G \times X \to G \underset{H}{\times} X$ be the canonical projection and let $V = q^{-1}p(U)$, which is open in $G \times X$. The restriction of the quotient mapping π to $\pi|V\colon V \to p^{-1}U$ is a quotient mapping - we leave it to the reader to verify. Hence it is enough to notice that ϕ is induced by a continuous mapping $V \to (G/GH) \times X$ defined by the formula $(g,x) \mapsto (gH, (s(gH)^{-1}g)x$. This mapping is obviously continuous, so it remains to check that it factors through $\underset{H}{\times} X$. To see this we take $h \in H$ and notice that

$$(gh^{-1}, hx) \mapsto (gh^{-1}H, (s(gh^{-1}H)^{-1}gh^{-1})hx = (gH, (s(gH)^{-1}g).$$

Next we check that the diagram commutes. This is a straightforward computation:

$$pr_1\phi([g,x]) = pr_1(gH, (s(gH)^{-1}g)x) = gH = p([g,x]).$$

This also shows that $\phi([g,x]) \in U \times X$, when $gH \in U$, so the range of ϕ is indeed $U \times X$.

Next we have to handle ψ. We let $W = \pi^{-1}U$ and consider the mapping $W \times X \to G \underset{H}{\times} X$ defined by $(g,x) \mapsto [s(gH),x]$. This is obviously continuous. Moreover when $gH = g'H$ (g,x) and (g',x) map to the same element. Hence we can quotient out and obtain a mapping $\psi\colon U \times X \to G \underset{H}{\times} X$. This mapping will be continuous, since factorization mapping $\pi \times \mathrm{id}$ is open and surjective, hence a quotient mapping.
Finally we see that $p(\psi(gH,x)) = p([s(gH),x]) = s(gH)H = gH \in U$, so the range of ψ does lie entirely in $p^{-1}U$.
It remains to show that ϕ and ψ are inverses of each other.

$$\phi(\psi(gH,x)) = \phi([s(gH),x]) = (s(gH)H, (s(gH)^{-1}s(gH))x) = (gH,x),$$

$$\psi(\phi([g,x])) = \psi(gH, (s(gH)^{-1}g)x) = [s(gH), (s(gH)^{-1}g)x] = [g,x],$$

since $s(gH) = gh$ for some $h \in H$ (exercise 1), so

$$[s(gH), (s(gH)^{-1}g)x] = [gh, h^{-1}x] = [g, x]$$

by the definition of the twisted product.

3. Suppose canonical projection $\pi\colon G \to G/H$ admits local cross-section. Suppose X is an H-space. Prove that canonical injection $i\colon X \to G \underset{H}{\times} X$, $i(x) = [e, x]$ is an embedding. (Hint: choose suitable local cross section and use the previous exercise.)

   Solution: By the proposition 1 we can assume that cross-section $s\colon U \to G$ is defined on the open neighbourhood U of eH and $s(eH) = e$.
   By the previous exercise there is an embedding $\psi\colon U \times X \to G \underset{H}{\times} X$ defined by $\psi(gH, x) = [s(gH), x]$. The restriction of ψ on $eH \times X$ is precisely i.

4. Suppose G is a topological group, H its closed subgroup and X is a G-space. Prove that the mapping $f\colon G \underset{H}{\times} X \to G/H \times X$ defined by

$$f([g, x]) = (gH, gx)$$

   is a G-homeomorphism. Here G acts on $G/H \times X$ componentwise, $g\cdot(g'H, x) = (gg'H, gx)$.

   Solution: Let us first check that f is well-defined. Suppose $h \in H$. Then

$$f([gh^{-1}, hx]) = (gh^{-1}H, gh^{-1}hx) = (gH, gx) = f([g, x]).$$

   Mapping f is clearly continuous. Let us define an inverse candidate for f, mapping $f'\colon G/H \times X \to G \underset{H}{\times} X$, by formula

$$f'(gH, x) = [g, g^{-1}x].$$

   f' is well-defined, since for any $h \in H$, if $g' = gh$, then

$$f'(g'H, x) = [gh, h^{-1}g^{-1}x] = [g, g^{-1}x] = f(gH, x).$$

   The fact that f' is continuous is seen as usual. The fact that f is G-equivariant is easily seen.
   It remains to show that f' is an inverse for f. This is a straightforward calculation:

$$f'(f([g, x])) = f'(gH, gx) = [g, g^{-1}gx] = [g, x],$$

$$f(f'(gH, x)) = f([g, g^{-1}x]) = (gH, g(g^{-1}x)) = (gH, x).$$

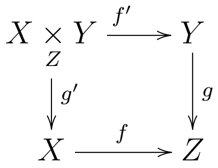
5. Suppose X, Y, Z are G-spaces, $f\colon X \to Z, g\colon Y \to Z$ G-equivariant continuous mappings. Define

$$X \underset{Z}{\times} Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subset X \times Y.$$

   The space $X \underset{Z}{\times} Y$ is called the *pull-back* of the pair (f, g). Restrictions of projections $X \times Y \to X$ and $X \times Y \to Y$ defines continuous mappings $g'\colon X \underset{Z}{\times} Y \to X$ and $f'\colon X \underset{Z}{\times} Y \to Y$.

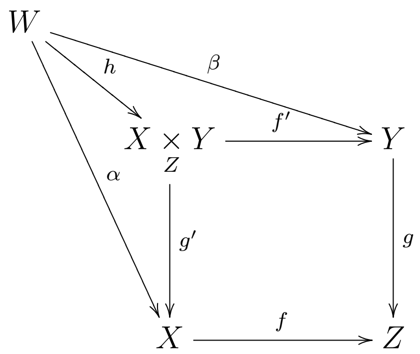
i) Prove that $X \underset{Z}{\times} Y$ is G-invariant subset of $X \times Y$, which has componentwise G-action, $g \cdot (x,y) = (gx, gy)$ and mappings f', g' are G-equivariant.

ii) Prove that $g \circ f' = g' \circ f$ i.e. the diagram



commutes.

iii) The push-out is *universal* with respect to such diagrams i.e. if W is a G-space, $\alpha\colon W \to X$, $\beta\colon W \to Y$ are G-equivariant mappings such that $g \circ \beta = \alpha \circ f$, then there exist unique G-equivariant mapping $h\colon W \to X \underset{Z}{\times} Y$ such that $f' \circ h = \beta$, $g' \circ h = \alpha$.
This is illustrated in the diagram below.



Solution: i) Suppose $h \in G$ and $(x,y) \in X \underset{Z}{\times} Y$. Then

$$f(hx) = hf(x) = hg(y) = g(hy),$$

so $h \cdot (x,y) = (hx, hy) \in X \underset{Z}{\times} Y$. Mapping f' is G-equivariant:

$$f'(h(x,y)) = f'(hx, hy) = hy = hf'(x,y).$$

The proof that g' is G-map is similar.

ii)

$$(g \circ f')(x,y) = g(y) = f(x) = (g' \circ f)(x,y).$$

iii)Suppose W is a G-space, $\alpha\colon W \to X$, $\beta\colon W \to Y$ are G-equivariant mappings such that $g \circ \beta = \alpha \circ f$. Suppose $h\colon W \to X \underset{Z}{\times} Y$ is such that $f' \circ h = \beta$, $g' \circ h = \alpha$. Then $h(w) = (a,b) \in X \underset{Z}{\times} \subset X \times Y$, where

$$a = pr_1 h(w) = g'(h(w)) = \alpha(w) \text{ and}$$
$$b = pr_2 h(w) = f'(h(w)) = \beta(w).$$

Hence we see that $h(w) = (\alpha(w), \beta(w))$ is uniquely determined. Conversely if we define h by this formula, h is clearly continuous G-mapping $W \to X \times Y$, so it only remains to show that the values of h lie in the subset $X \underset{Z}{\times} Y$. But this follows precisely from the condition $g \circ \beta = \alpha \circ f$.

6. Here $X, Y, Z, X \underset{Z}{\times} Y, f, g, f', g'$ are as in the exercise 5. above. Prove the following claims.
i) If f is surjective, f' is also surjective.
ii) If f is injective, f' is also injective.
iii) If f is open, f' is open.

Solution: i) Suppose f is surjective. Let $y \in Y$. Since f is surjection, there exists $x \in X$ such that $f(x) = g(y)$. Now the pair (x, y) belongs to $X \underset{Z}{\times} Y$ and $f'(x, y) = y$.

ii) Suppose f is an injection. Suppose $f'(x, y) = y = y' = f'(x,', y'$. Then $y = y'$. But since $(x, y), (x', y) \in X \underset{Z}{\times} Y$,

$$f(x) = g(y) = f(x').$$

By the injectivity of f this implies that $x = x'$. Hence $(x, y) = (x', y')$.

iii) Suppose f is open. Let $W \subset X \underset{Z}{\times} Y$ be open. We need to prove that $f'(W)$ is open in Y. Since W is open in the relative topology, there exists V open in $X \times Y$ such that $W = V \cap (X \underset{Z}{\times} Y)$.
Suppose $y \in f'(W)$, then there exists $x \in X$ such that $(x, y) \in W$. Then $f(x) = g(y)$ and $(x, y) \in V$, hence there exists neighbourhood U, U' of x and y in X and Y respectively such that $U \times U' \subset V$.
By assumption $U'' = U' \cap g^{-1}f(U)$ is a neighbourhood of y in Y. Enough to show that $U'' \subset f'(W)$. Suppose $y' \in U' \cap g^{-1}f(U)$. Then $y' \in U'$ and $g(y) = f(x')$ for some $x' \in U$. We see that $(x', y') \in (U \times U') \cap X \underset{Z}{\times} Y \subset W$ and $f'(x', y') = y'$. Thus $U'' \subset f'(W)$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.