Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 12
23.04-27.04.2012

1. Suppose G is a topological group and H closed subgroup of G. A local cross-section of canonical projection $\pi\colon G \to G/H$ is a continuous mapping $s\colon U \to G$ defined on some open non-empty subset U of G/H such that $\pi(s(x)) = x$ for all $x \in U$.

   If there exists at least one local cross-section of $\pi\colon G \to G/H$ we say that canonical projection π admits local cross-section.

   a) Suppose $\pi\colon G \to G/H$ admits local cross-section. Prove that every point $x \in G/H$ has an open neighbourhood U such that there exists local cross section $s\colon U \to G$ of π.

   b) Suppose $s\colon U \to G$ is a local cross-section of π, where U is open subset of G/H. Suppose $g \in \pi^{-1}(U)$ and let $g' = s(\pi(g))$. Prove that $gH = g'H$.

   c) Suppose $\pi\colon G \to G/H$ admits local cross-section. Let $g \in G$. Prove that there exists a local cross section $s\colon U \to G$ of π such that $g \in \pi^{-1}U$ and $s(gH) = g$.

2. Suppose G is a topological group, H is its closed subgroups and suppose projection $\pi\colon G \to G/H$ admits local cross-section. Suppose X is an H-space and let $p\colon G \times_H X \to G/H$, $p([g,x]) = gH$.
   Let $s\colon U \to G$ be a local cross section of π.
   Define $\phi\colon p^{-1}U \to U \times X$, $\phi([g,x]) = (gH, (s(gH)^{-1}g)x)$, $\psi\colon U \times X \to p^{-1}U$, $\psi(gH, x) = [s(gH), x]$. Prove that ϕ and ψ are well-defined continuous G-mappings and inverses of each other (hence G-homeomorphisms) and the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times X \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array}$$

   commutes.

3. Suppose canonical projection $\pi\colon G \to G/H$ admits local cross-section. Suppose X is an H-space. Prove that canonical injection $i\colon X \to G \times_H X$, $i(x) = [e, x]$ is an embedding. (Hint: choose suitable local cross section and use the previous exercise.)

4. Suppose G is a topological group, H its closed subgroup and X is a G-space. Prove that the mapping $f\colon G \underset{H}{\times} X \to G/H \times X$ defined by

$$f([g,x]) = (gH, gx)$$

is a G-homeomorphism. Here G acts on $G/H \times X$ componentwise, $g\cdot(g'H,x) = (gg'H, gx)$.

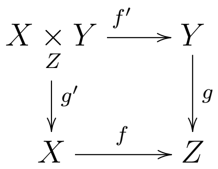
5. Suppose X, Y, Z are G-spaces, $f\colon X \to Z, g\colon Y \to Z$ G-equivariant continuous mappings. Define

$$X \underset{Z}{\times} Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\} \subset X \times Y.$$

The space $X \underset{Z}{\times} Y$ is called the *pull-back* of the pair (f,g). Restrictions of projections $X \times Y \to X$ and $X \times Y \to Y$ defines continuous mappings $g'\colon X \underset{Z}{\times} Y \to X$ and $f'\colon X \underset{Z}{\times} Y \to Y$.

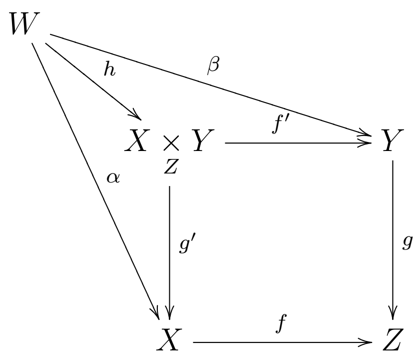
i) Prove that $X \underset{Z}{\times} Y$ is G-invariant subset of $X \times Y$, which has componentwise G-action, $g \cdot (x,y) = (gx, gy)$ and mappings f', g' are G-equivariant.

ii) Prove that $g \circ f' = g' \circ f$ i.e. the diagram



commutes.

iii) The push-out is *universal* with respect to such diagrams i.e. if W is a G-space, $\alpha\colon W \to X$, $\beta\colon W \to Y$ are G-equivariant mappings such that $g \circ \beta = \alpha \circ f$, then there exist unique G-equivariant mapping $h\colon W \to X \underset{Z}{\times} Y$ such that $f' \circ h = \beta$, $g' \circ h = \alpha$.
This is illustrated in the diagram below.



6. Here $X, Y, Z, X \underset{Z}{\times} Y, f, g, f', g'$ are as in the exercise 5. above. Prove the following claims.
i) If f is surjective, f' is also surjective.
ii) If f is injective, f' is also injective.
iii) If f is open, f' is open.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.