Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 10
Solutions

1. Suppose G is locally compact and H is a closed subgroup of G. Prove that G/H is Borel proper G-space if and only if H is compact.

   Solution: If G/H is Borel proper, then G_x is compact for every $x \in G/H$ (Top. Trans. Groups II, Proposition 4.13 a)). In particular it is true for $x = eH$. But as we already know (exercise 4.3) that $G_{eH} = H$. Hence H must be compact.

   Conversely suppose H is compact. Then canonical projection $\pi\colon G \to G/H$ is proper, i.e. $\pi^{-1}K$ is compact in G for every compact $K \subset G/H$ (Top. Trans. Groups I, Theorem 1.11).
   Let $K \subset G/H$ be compact. We need to show that $G(K|K)$ is compact. Suppose $g \in G(K|K)$, then there exist $xH, yH \in K$ such that $gxH = yH$. Now $x, y \in \pi^{-1}K = A$ (which is compact) and $gx \in yH$ i.e. $g \in AHA^{-1}$. Hence $G(K|K) \subset AHA^{-1}$, which is compact. Since $G(K|K)$ is closed (Top. Trans. Groups II, Lemma 4.4.), it follows that $G(K|K)$ is compact.

2. Consider action of $\mathbb{R}$ on $X = \mathbb{R}^2 \setminus \{0\}$ defined by
$$t(x, y) = (e^t x, e^{-t} y).$$
   i) Prove that every point $z \in X$ has a neighbourhood U such that $\overline{G(U|U)}$ is compact (Hint: look at the small ball neighbourhoods).
   ii) Let $z = (1, 0)$ and $v = (0, 1)$. Prove that for all neighbourhoods U of z and V of v the set $G(U|V)$ is unbounded, hence not relatively compact.
   iii) Prove that X/G is not Hausdorff.

   Solution: i) Since a subset of real line is relatively compact if and only if it is bounded, enough to find U such that $G(U|U)$ is bounded. Let $z = (x, y) \in X$ be arbitrary. Then either $x \neq 0$ or $y \neq 0$. We go through the case $x \neq 0$, case $y \neq 0$ is similar.
   Choose $a, b > 0$ such that $a < |x| < b$ and let U be a neighbourhood of x such that $a < |x'| < b$ for all $(x', y') \in U$. Suppose $t \in G(U|U)$, then $(e^t x', e^{-t} y') \in U$ for some $(x', y') \in U$, so in particular $a < e^t|x'| < b$, which implies that
$$a/b < a/|x'| < e^t < b/|x'| < b/a,$$
   so $t \in]\ln(a/b), \ln(b/a)[$. Hence $G(U|U)$ is bounded and we are done.

   ii) Suppose $\varepsilon > 0$ is arbitrary. It is enough to show that for $U = B(z, \varepsilon)$ and $V = B(v, \varepsilon)$ the set $G(U|V)$ is unbounded. For $a \in]0, \varepsilon[$ the point $(1, a)$ is in U and $t \cdot (1, a) = (e^t, e^{-t} a) \in V$ for $t = \ln(a)$. Hence $G(U|V)$ includes all numbers of the form $ln(a), a \in]0, \varepsilon[$ and this set is unbounded from below.

iii) By i) X is Cartan and by ii) X is not Koszul. Hence by Proposition 1.5 in "More on Proper actions"X/G is not Hausdorff. Alternatively ii) easily implies that points $\mathbb{R}z$ and $\mathbb{R}v$ are different points in X/G which don't have disjoint neighbourhoods.

3. Hausdorff space X is called **compactly generated** if the following condition is satisfied:
Suppose $F \subset X$ is such that $F \cap K$ is closed in K for all compact $K \subset X$. Then F is closed.
a) Prove that X is compactly generated if and only if the following condition is satisfied: Suppose $U \subset X$ is such that $U \cap K$ is open in K for all compact $K \subset X$. Then U is open.
b) Prove that every locally compact Hausdorff space is compactly generated (Hint: use a)).
c) Prove that every first-countable space Hausdorff space, in particular every metric space, is compactly generated. Reminder: space is first-countable if every point has a countable neighbourhood base. In first-countable spaces sequences are sufficient to describe eg. closures. (Hint: use the fact that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to $x \in X$, then the set $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is compact.)

Solution: a) Suppose $A \subset X$ and K is a compact subset of X. Then $A \cap K$ is open in K if and only if $(K \setminus A) \cap K = K \setminus (A \cap K)$ is closed in K. The claim follows.

b) Suppose $U \subset X$ is such that $U \cap K$ is open in K for all compact $K \subset X$. Suppose $x \in U$ and let V be a neighbourhood of x such that $\bar{V}$ is compact. Our assumption implies that $U \cap \bar{V}$ is open in $\bar{V}$. By the properties of relative topology this implies that $U \cap V = (U \cap \bar{V}) \cap V$ is open in V. Since V itself is open in X, by transitivity of relative topology $U \cap V$ is open in X. Hence every point x of U has a neighbourhood $V' \subset V$. Thus V is open in X (for instance since it can be written as a union of open sets U').

c) Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in a topological space X that converges to $x \in X$. We first prove that then the set $K = \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is compact. Suppose $(U_\alpha)_{\alpha\in\mathcal{A}}$ is an open covering of K. Then $x \in U_\alpha$ for some $\alpha \in \mathcal{A}$. Since $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U_\alpha$ for $n \geq n_0$. For every $n < n_0$ choose an index α_n such that $x_n \in U_{\alpha_n}$. Then $\{U_{\alpha_n} \mid n < n_0\} \cup \{U_\alpha\}$ is a finite subcover of $(U_\alpha)_{\alpha\in\mathcal{A}}$. Hence K is compact.

Now suppose X is a first-countable Hausdorff space and let $F \subset X$ be such that $F \cap K$ is closed in K for all compact $K \subset X$. We need to show F is closed. Suppose $y \in \bar{F}$. Since X is first-countable there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in F such that $x_n \to y$. Let

$$K = \{x_n \mid n \in \mathbb{N}\} \cup \{y\},$$

then K is compact (see above), hence $F \cap K$ is closed in K. But

$$F \cap K = \{x_n \mid n \in \mathbb{N}\}$$

and $y \in K$ is certainly in the closure of this set. Hence $y \in F \cap K \subset F$ and we are done.

4. Suppose X and Y are Hausdroff spaces, and $f\colon X \to Y$ is proper i.e. $f^{-1}K$ is compact for every compact $K \subset Y$. Assume that Y is compactly generated. Show that f is a closed mapping. (Hint: enough to prove that $f(C) \cap K$ is closed for every compact $K \subset Y$ and closed $C \subset X$. But $f(C) \cap K = f|f^{-1}(K)(C \cap f^{-1}(K))$.)

   Solution: Suppose C is closed in X. We want to prove that $f(C)$ is closed in Y. Since Y is compactly generated, it is enough to prove that $f(C) \cap K$ is closed for every compact $K \subset Y$. But
   $$f(C) \cap K = f|f^{-1}(K)(C \cap f^{-1}(K)).$$
   where $f|f^{-1}(K)\colon f^{-1}(K) \to Y$ is a continuous mapping from compact space $f^{-1}(K)$ to a Hausdorff space Y. Such a mapping is always closed (Topology II). Since $C \cap f^{-1}(K)$ is closed in X (intersection of two closed sets), it follows that $f(C) \cap K$ is closed. We are done.

5. Suppose X, Y, Z are Hausdorff spaces, $f\colon X \to Y$ and $h\colon Y \to Z$ continuous. Suppose $h \circ f$ is proper. Prove that
   i) f is proper.
   ii) if f is surjection, then h is proper.

   Solution: i) Suppose $K \subset Y$ is compact. Then $L = h(K)$ is compact in Z. Since $h \circ f$ is proper it follows that $f^{-1}h^{-1}L = (h \circ f)^{-1}L$ is compact. But $K \subset h^{-1}L$, so $f^{-1}K \subset f^{-1}h^{-1}L$. Since f is continuous and K is closed, $f^{-1}K$ is closed. Hence $f^{-1}K$ is a closed subset of a compact set, i.e. compact itself.

   ii) Suppose f is a surjection. Let $K \subset Z$ be compact. Since $h \circ f$ is proper, $L = f^{-1}h^{-1}K = (h \circ f)^{-1}K$ is compact. But f is surjection, so $f(f^{-1}A) = A$ for all subsets $A \subset Y$, thus
   $$h^{-1}K = f(L)$$
   is compact as a continuous image of a compact set.

6. Suppose X is a G-space and U, V open subsets of X. Prove that $G(U|V)$ is open in G.

   Solution: Suppose $g \in G(U|V)$, then $gU \cap V \neq \emptyset$. Let $u \in U$ be such that $gu \in V$. Since action of G in X is continuous and V is open, there exist neighbourhood A of g in G and B of u in X such that $AB \subset V$. In particular $g'u \in V$ for all $g' \in A$. This implies that $A \subset G(U|V)$. The claim follows.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.