Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 1
23-29.01.2012

1. Prove that $\mathbb{R}^n$, $n \in \mathbb{N}$ equipped with standard topology and addition of vectors
$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
is a topological group.

2. Consider $\mathbb{R}^2$ as the set of complex numbers $\mathbb{C}$, equipped with the standard multiplication of complex numbers defined by
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$
Show that $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a topological group, if equipped with this multiplucation and standard topology as a subset of $\mathbb{R}^2$. Why did we have to exclude 0?

3. Topological space X is called **homogeneous** if for every pair of points $x, y \in X$ there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$. Prove that every topological group is homogeneous as a topological space.
Conclude that the unit interval $I = [0, 1]$ (equipped with its standard topology) cannot be given a structure of a topological group.

4. a) Suppose G is a topological group, U an open neighbourhood of the neutral element $e \in G$ and W an open neighbourhood of e such that
$$WW^{-1} \subset U.$$
Prove that $\overline{W} \subset U$. (Hint: xW is a neighbourhood of any point $x \in \overline{W}$).
b) Topological space X is called **regular** if for any $x \in X$ and any open neighbourhood U of x there exists a neighbourhood W of x such that $\overline{W} \subset U$. Prove that every topological group is regular as a topological space.

5. Suppose X is a T_1 and regular topological space. Prove that X is Hausdorff. Conclude that every topological group is Hausdorff as a topological space.

6. A topological space is said to be T_0-space, if for every pair of points x, y there exists a neighbourhood U of one of the points, that do not contain the other point.
a) Suppose X is T_0 and regular. Prove that X is Hausdorff.
b) Suppose $(G, \cdot, \tau)$ is a triple, where $(G, \cdot)$ is a group and τ is a topology in the set G. Denote by $f\colon G \times G \to G$ the mapping defined by
$$f(x, y) = xy^{-1}.$$
Prove that G is a topological group if and only if G is T_0 as a topological space and f is continuous.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.