1 Cartan and Koszul spaces

Suppose G is a locally compact topological group and X is a Hausdorff G-space. We say that X is a *Cartan* G-space if every $x \in X$ has a neighbourhood U such that $G(U|U)$ is relatively compact i.e. $\overline{G(U|U)}$ is a compact subset of G.

Proposition 1.1. *Suppose X is a Cartan G-space. Then*
i) G_x is compact for every $x \in X$.
ii) Every orbit Gx is closed in X.
iii) Canonical bijection $G/G_x \to Gx$, $gG_x \mapsto gx$ is a homeomorphism for every $x \in X$.
iv) X/G is T_1.

Proof. ii) and iv) are clearly equivalent. To prove ii) and iii) it is enough to show that the mapping $\phi_x \colon G \to X$ defined by $g \mapsto gx$ is a closed mapping for every $x \in X$ - since then $Gx = \phi_x(G)$ is closed and $\phi_x \colon G \to Gx$ is a closed surjection, hence quotient mapping, so the induced mapping in iii) must be homeomorphism.

Suppose $C \subset G$ is closed. To prove $\phi_x(C)$ is closed suppose $y \in \overline{\phi_x(C)}$. Then there exists net g_α in C such that $g_\alpha x$ converges to y in X. Let U be a neighbourhood of y such that $\overline{G(U|U)}$ is compact. Since $g_\alpha x$ converges to y, there exists α_0 such that $g_\alpha x \in U$ for all $\alpha \geq \alpha_0$. Now

$$g_\alpha x = (g_\alpha g_{\alpha_0}^{-1})(g_{\alpha_0} x) \in U \cap g_\alpha g_{\alpha_0}^{-1} U,$$

so $g_\alpha g_{\alpha_0}^{-1} \in G(U|U)$. By restricting net g_α to the cofinal subset $\{\alpha \geq \alpha_0\}$, which is a directed set, we see that $g_\alpha g_{\alpha_0}^{-1}$ is a net in the compact set $\overline{G(U|U)}$, hence it has a convergent subnet. Multiplying by g_{α_0} we see that g_α has a convergent subnet. Hence we may assume $g_\alpha \mapsto g$, which must be an element of C, since C is closed. Now $g_\alpha x$ converges to gx, so by uniqueness of the limit in Hausdorff spaces $y = gx \in \phi_x(C)$. The claim is proved.

It only remains to show that G_x is compact for every $x \in X$. Suppose U is a neighbourhood of X such that $\overline{G(U|U)}$ is compact. Clearly $G_x \subset G(U|U)$, so it is compact as a closed subset of a compact space. □

Example 1.2. *Suppose G is locally compact and H its closed subgroup. Then homogeneous space G/H is a Cartan G-space if and only if H is compact (exercise).*

Example 1.3. $\mathbb{R}$ *acts on* $\mathbb{R}^2 \setminus \{0\}$ *by*

$$t(x, y) = (e^t x, e^{-t} y).$$

This action is Cartan, but the orbit space is not Hausdorff (exercise). Since $\mathbb{R}^2 \setminus \{0\}$ is locally compact, Proposition 4.14 in [3] implies that this action is also not Borel proper.

Thus we see that Cartan assumption is not enough to prove that orbit space is Hausdorff, so we need a stronger assumption to guarantee that.

Definition 1.4. *Suppose X is a Hausdorff G-space, where G is locally compact group. We say that X is* Koszul*-space if for all $x, y \in X$ there exists a neighbourhood U of x and a neigyhbourhood V of y such that $\overline{G(U|V)}$ is compact.*

Clearly every Koszul G-space is a Cartan G-space. The converse is not true, as the next proposition and example 1.3 above show.

Proposition 1.5. *Suppose X is a Hausdorff G-space, where G is locally compact group. Then X is Koszul if and only if X is Cartan and X/G is Hausdorff.*

Proof. Suppose X is Koszul. We need to show that X/G is Hausdorff. From general topology it is known that a topological space Y is Hausdorff if and only if the diagonal

$$\Delta = \{(y, y) \in Y \times Y \mid y \in Y\}$$

is closed in the product space $Y \times Y$. Let $\pi\colon X \to X/G$ be the canonical projection. Then $\pi \times \pi\colon X \times X \to X/G \times X/G$ is an open surjective mapping, in particular a quotient mapping, so the diagonal $\Delta \subset X/G \times X/G$ is closed if and only if so-called graph of the relation $x \sim gx, g \in G, x \in X$, which is exactly the set

$$\Gamma = \{(x, gx) \mid x \in X, g \in G\} = (\pi \times \pi)^{-1}(\Delta),$$

is closed in $X \times X$. Thus it is enough to show that Γ is closed.
Suppose $(x_\alpha, g_\alpha x_\alpha)$ is a net in Γ converging to $(x, y) \in X \times X$. We need to show that $(x, y) \in \Gamma$.
Now x_α converges to x and $g_\alpha x_\alpha$ converges to y. Let U and V be neighbourhoods of x and y such that $\overline{G(U|V)}$ is compact. There exists α_0 such that $x_\alpha \in U$ and $g_\alpha x_\alpha \in V$ for $\alpha \geq \alpha_0$. Hence $g_\alpha \in \overline{G(U|V)}$ for all $\alpha \geq \alpha_0$, and since $\overline{G(U|V)}$ is compact we may assume that g_α converges to $g \in G$. Hence $g_\alpha x_\alpha$ converges to gx, i.e. $y = gx$ and $(x, y) \in \Gamma$.

Suppose conversely X is Cartan and X/G is Hausdorff. Let $x, y \in X$. If $y = gx$ for some $g \in G$ let U be a neighbourhood of x such that $\overline{G(U|U)}$ is compact. Then $V = gU$ is a neighbourhood of y such that $\overline{G(U|V)} = g^{-1}\overline{G(U|U)}$ is compact. Otherwise $\pi(x) \neq \pi(y)$, so there are disjoint neighbourhoods W, W' of $\pi(x)$ and $\pi(y)$. Now $U = \pi^{-1}W$ and $V = \pi^{-1}W'$ are disjoint G-neighbourhoods of x and y, so $G(U|V) = \emptyset$. □

Notice that in fact we have proved that every Koszul space has property stronger than the one mentioned in its definition - if x and y are in the different orbits, then they have neighbourhoods U and V such that $G(U|V)$ is even empty.

Proposition 1.6. *Every Koszul G-space is Borel proper.*

Proof. Suppose $A \subset X$ is compact. Fix $a \in A$. For every $x \in A$ there exists a neighbourhood U_a of a and a neighbourhood V_a of x such that $\overline{G(U_a|V_a)}$ is compact. Take a finite subcover $V_1, \dots, V_n$ of A, and let $U_1, \dots, U_n$ be corresponding neighbourhoods of x. Then $U = \bigcap_{i=1}^n U_i$ is a neighbourhood of x and

$$G(U|A) \subset \bigcup_{i=}^{n} G(U|V_i) \subset \bigcup_{i=1}^{n} G(U_i|V_i).$$

Thus we have shown that every $x \in A$ has a neighbourhood U such that $G(U|A)$ is relatively compact. Now take such a neighbourhood for every $x \in X$ and substruct a finite subcover $U_1, \dots, U_n$ of A. Then

$$G(A|A) \subset \bigcup_{i=1}^{n} G(U_i|A),$$

so $G(A|A)$ is relatively compact. But on the other hand $G(A|A)$ is closed by Lemma 4.4. in [3]. □

Remark 1.7. *We have defined both notions of Cartan and Koszul spaces for the actions of locally compact groups only. The reader might notice that we did not use the local compactness of G directly anyway, so she might wonder why these assumptions is made.*
However it is easy to see that both definitions make sense for locally compact groups only. In fact one can easily verify that if U is an open non-empty subset of G-space X, then $G(U|U)$ is an open neighbourhood of the neutral element $e \in G$, so if X is a G-space, for which $G(U|U)$ is relatively compact, G **must be locally compact.**

The following result allows one to generalize the notion of Koszul space for non-locally compact groups.

Lemma 1.8. *Suppose G is a locally compact group acting on Hausdorff space X. Then X is Koszul if and only if the mapping $\phi\colon G \times X \to X \times X$, $\phi(g, x) = (gx, x)$ is proper in the following sense:*
ϕ is closed and $\phi^{-1}(z)$ is compact for every $z \in X \times X$.

Proof. Omitted. Se [1, 2.3.6] or [2, Proposition 3.21] □

Using this result we can define the notion of Koszul space to the actions of general groups - one says that action is Koszul (or simply *proper*) if the mapping ϕ defined as above is proper in the sense mentioned in the lemma above. One can prove that with this notion properties in propositions 1.1 and 1.5 are true in general. For details and proofs see [2, I.3].

2 Palais proper spaces

We have seen that Koszul and even Cartan spaces have some nice properties, but one gets much more theoretical results if one considers certain stronger notion of properness, which was first defined by Palais in his paper [4].
Suppose X is a Hausdorff G-space, where G is a locally compact group. We say that a subset A of X is *small* if every point $x \in X$ has a neighbourhood U in X such that $G(A|U)$ is relatively compact.

Lemma 2.1. *i) A subset of a small set is small.*
ii) Finite union of small sets is small.
iii) Suppose $A \subset X$ is small and $C \subset X$ is compact. Then $G(A|C)$ is relatively compact.

Proof. i) If $B \subset A$, then $G(B|U) \subset G(A|U)$, so the claim follows.
ii) Suppose $A_1, \dots, A_n$ are small. Let $x \in X$, then x has neighbourhoods $U_1, \dots, U_n$ such that $G(A_i|U_i)$ is relatively compact. Now $V = \bigcap\{i = 1\}^n U_i$ is a neighbourhood of x and

$$G(\bigcup_{i=1}^{n} A_i|V) \subset \bigcup_{i=1}^{n} G(A_i|U_i),$$

hence $G(\bigcup_{i=1}^{n} A_i|V)$ is relatively compact.
iii) Choose for any $c \in C$ a neighbourhood U_c of c such that $G(A|U_c)$ is relatively compact. Then choose a finite cover $U_{c_1}, \dots, U_{c_n}$ of C. Since

$$G(A|C) \subset \bigcup_{i=1}^{n} G(A|U_i),$$

the claim follows. □

Lemma 2.2. *Suppose $f\colon X \to Y$ is a G-map between G-spaces. Then for all $A, B \subset X$ we have*

$$G(A|B) \subset G(fA|fB).$$

For all $A, B \subset Y$ we have

$$G(f^{-1}A|f^{-1}B) \subset G(A|B).$$

If $A \subset Y$ is small, then $f^{-1}A$ is small.

Proof. Suppose $g \in G(A|B)$. Then $ga \in B$ for some $a \in A$. Hence $gf(a) = f(ga) \in fB$, so $g \in G(fA|fB)$.
Since $f(f^{-1}C) \subset C$ for all $C \subset Y$, this result implies that

$$G(f^{-1}A|f^{-1}B) \subset G(f(f^{-1}A)|f(f^{-1}B)) \subset G(A|B).$$

Now suppose $A \subset Y$ is small. Suppose $x \in X$, then $y = f(x)$ has a neighbourhood V such that $G(A|V)$ is relatively compact. Since $U = f^{-1}U$ is a neighbourhood of x and

$$G(f^{-1}A|U) \subset G(A|V),$$

it follows that $f^{-1}A$ is small. $\square$

Definition 2.3. *Suppose G and X are as above. We say X is a* Palais proper *G-space if every $x \in X$ has a small neighbourhood.*

Proposition 2.4. *Suppose X is a Palais proper G-space. Then*
i) Every compact subset of X is small.
ii) X is Cartan, Koszul and Borel proper.
iii) For every $x \in X$ G_x is compact, orbit Gx is closed in X, X/G is Hausdorff and canonical mapping $G/G_x \to Gx$, $gG_x \mapsto gx$ is a homeomorphism.
iv) Suppose Y is a G-space and there exists G-equivariant $f\colon Y \to X$. Then Y is Palais proper.
v) Any G-subspace of X is Palais proper.

Proof. i) Suppose $K \subset X$ is compact. Then K can be covered by small neighbourhoods of its points, so choosing a finite subcover we see that K is a subset of finite union of small sets, which is small by 2.1, i) and ii).

ii) and iii) - by previous results it is enough to prove that X is Koszul. This follows directly from definition - if $x, y \in X$ let U be a small neighbourhood of x and V be a neighbourhood of y such that $G(U|V)$ is relatively compact. The existence of these neighbourhoods proves that X is Koszul.

iv) Suppose $y \in Y$ and U is a small neighbourhood of $x = f(y)$. Then, by Lemma 2.2, $f^{-1}(U)$ is a small neighbourhood of y.

v) Suppose Y is a G-subspace of X. Then inclusion $i\colon Y \to X$ is a G-mapping, so the claim follows from iv).

$\square$

Just as Koszul is the same as Cartan+Hausdorff orbit space, we can characterise Palais proper spaces as Cartan spaces with regular orbit space, at least when the space itself is regular.

Proposition 2.5. *Suppose X is a regular G-space. Then X is Palais proper if and only if X is Cartan and X/G is regular.*

Proof. Suppose X is Palais proper regular G-space. Let us prove that X/G is regular. By the previous proposition X/G is Hausdorff. Suppose $y = \pi(x) \in X/G$ and V is a neighbourhood of y. We need to find a neighbourhood U of y such that $\overline{U} \subset V$.
Now $\pi^{-1}V$ is a neighbourhood of x. Since X is regular and Palais proper there exists

a neighbourhood W of x such that $\overline{W} \subset \pi^{-1}V$ and $\overline{W}$ is small. We claim that $G\overline{W}$ is closed in X. Suppose $g_\alpha w_\alpha$ is a net in $G\overline{W}$ converging to $x \in X$. Let A be a neighbourhood of x such that $G(\overline{W}|A)$ is relatively compact (which exists since $\overline{W}$ is small). Switching to a subnet if necessary we may assume that $g_\alpha w_\alpha \in A$ for all α. This implies that $g_\alpha \in G(\overline{W}|A)$ for all α, so by switching to a subnet again we may assume that $g_\alpha \to g \in G$. Hence $w_\alpha = (g_\alpha)^{-1} g_\alpha w_\alpha \to g^{-1}x$, so $g^{-1}x \in \overline{W}$ and hence $x = g(g^{-1}x) \in G\overline{W}$.
Hence $G\overline{W}$ is closed in X. On the other hand $G\overline{W} = \pi^{-1}\pi(\overline{W})$, hence $\pi(\overline{W})$ is closed in X/G. Thus

$$p(W) \subset \overline{p(W)} \subset \pi(\overline{W}) \subset U$$

and $p(W)$ is a neighbourhood of y, since π is open. This proves regularity of X/G.

Conversely suppose X is a Cartan space and X/G is regular. Let $x \in X$ and suppose U is a neighbourhood of x such that $G(U|U)$ is relatively compact. Now $\pi(U)$ is neighbourhood of $\pi(x)$, so there is a neigbourhood V of $\pi(x)$ such that $\overline{V} \subset \pi(U)$. Let $W = \pi^{-1}V$, then W is a G-neighbourhood of Gx and $\overline{W} \subset \pi^{-1}U = GU$.
Let $O = V \cap W$, then O is neighbourhood of x. We claim that O is small. Suppose $y \in X$. If $y \in GU$, then gU is a neighbourhood of y for some $g \in G$ and $G(U|gU) = g^{-1}G(U|U)$ is relatively compact. If $y \notin GU$ then $X \setminus \overline{W}$ is a neighbourhood of y and $G(O|X \setminus \overline{W}) = \emptyset$, since $X \setminus \overline{W}$ is a G-subset, which does not intersect O. □

Remark 2.6. *Actually even more powerful and important result is true - if X is* **completely regular** *G-space, then X is Palais proper if and only if X is Cartan and X/G is completely regular. Recall that a Hausdorff space is called completely regular if for every $x \in X$ and a neighbourhood U of x there exists continuous $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \notin U$. Many topologists believe that completely regular spaces form "the most suitable" class of topological spaces and it has rich theory and a lot of applications, which more general classes of spaces lack. Interested reader will find a proof of the claim above in the paper [4].*

Although Cartan spaces are much more general than Palais proper spaces, every Cartan space is locally Palais proper.

Lemma 2.7. *Suppose X is Cartan G-space. Then every $x \in X$ has a G-neighbourhood U, such that U is Palais proper. In particular X/G is locally Hausdorff and if X is regular (or completely regular) X/G is even locally (completely) regular i.e. every point $y \in X/G$ has a neighbourhood, which is Hausdorff/regular (completely regular).*

Proof. Let V be a neighbourhood of x such that $G(V|V)$ is relatively compact and let $U = GV$, then U is a G-neighbourhood of x. We claim that U is Palais proper as a G-space. Suppose $y, z \in GU$, then $y \in gU, z \in hU$ for some $g, h \in G$. Since $G(gU|hU)$ is relatively compact, it follows that every point has a small neighbourhood (of the form gU) in U.

Now $\pi(U)$ is a neighbourhood of $y = \pi(x)$ in X/G, since π is open, and $\pi|U\colon U \to \pi(U)$ is an open surjection, hence induces a homeomorphism $U/G \to \pi(U)$. Since U/G is

Hausdorff / (completely) regular by the previous proposition, it follows that every point in X/G has a Hausdorff regular (even completely regular - see remark above) neighbourhood . $\square$

If the G-space X is locally compact, all "properness" notions (except Cartan) coincide. In particular this is true if X is a manifold.

Theorem 2.8. *Suppose X is a locally compact G-space, where G is locally compact. Then the following are equivalent.*
i) X is Palais proper.
ii) X is Koszul.
iii) X is Borel proper.

Proof. We have already seen that i) implies ii) and ii) implies iii). Suppose X is Borel proper. Suppose $x, y \in X$. Then x and y have relatively compact neighbourhoods U and V. Since X is Borel proper $G(U|V)$ is relatively compact. It follows that in particular x has a small neighbourhood (U does the trick). $\square$

Remark 2.9. *If X is not locally compact, previous proposition is not necessary true. There are examples of G-spaces, which are Borel proper, but not even Cartan, with non-Hausdorff orbit spaces, and there are examples of Koszul spaces, which are not Palais proper.*

Corollary 2.10. *Suppose X is a locally compact Hausdorff Borel proper G-space, where G is locally compact. Then X/G is Hausdorff and locally compact, hence also regular.*
(compare this with Proposition 4.14 in [3])

Proof. Since X is Palais proper by the previous theorem, the claim follows from 2.4 or 2.5. In fact in this case it is enough to verify that X/G is Hausdorff, since then it is locally compact (as an image $\pi(X)$, where π is open and continuous) and every locally compact Hausdorff space is regular. $\square$

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