0.1 Bounded and Compact operators

For a given inner-product space V, a linear operator $T : V \to V$ is called *bounded* (with respect to a given norm) if there exists a constant C such that for all v, $||T(v)|| \le C||v||$. In this case the smallest such C is denoted $|T|$. It is easy to see from the definitions that the following holds:

$$|T| = \sup_{0 \ne v \in V} \left(\frac{||T(v)||}{||v||} \right).$$

A bounded operator T is said to be *compact* if for a sequence $v_1, v_2, \ldots$ such that $||v_i|| \le 1 \; \forall i$, the sequence $T(v_1), T(v_2), \ldots,$ has a convergent subsequence.

T is said to be *symmetric* if $\langle T(v), w\rangle = \langle v, T(w)\rangle$.

Lemma 1. *If $T : V \to V$ is a bounded symmetric compact operator then:*

$$|T| = \sup_{0 \ne v \in V} \left(\frac{\langle T(v), v\rangle}{\langle v, v\rangle} \right).$$

Proof. Let B denote the right-hand side of the given equation. By the Schwartz inequality we get that for any $v \ne 0$:

$$|\langle T(v), v\rangle| \le ||T(v)|| \; ||v|| \le |T| \; ||v||^2 = |T|\langle v, v\rangle,$$

which gives us

$$\frac{|\langle T(v), v\rangle|}{\langle v, v\rangle} \le |T|,$$

and so $B \le |T|$.

On the other hand, we assume $T(v) \ne 0$ and k is some positive constant, since we know that T is symmetric we have:

$$\begin{aligned}\langle T(kv + k^{-1}T(v)), kv + k^{-1}T(v)\rangle &= \langle T(kv), kv\rangle + 2\langle T(kv), k^{-1}T(v)\rangle + \langle T(k^{-1}T(v)), k^{-1}T(v)\rangle \\ &= \langle T(kv), kv\rangle + 2\langle T(v), T(v)\rangle + \langle T(k^{-1}T(v)), k^{-1}T(v)\rangle\end{aligned}$$

$$\langle T(kv - k^{-1}T(v)), kv - k^{-1}T(v)\rangle = \langle T(kv), kv\rangle - 2\langle T(v), T(v)\rangle + \langle T(k^{-1}T(v)), k^{-1}T(v)\rangle.$$

By definition, for all $v \ne 0$ we have that $\frac{|\langle T(v), v\rangle|}{\langle v, v\rangle} \le B$ and so $|\langle T(v), v\rangle| \le B \; \langle v, v\rangle$. Combining the equations above we get

$$\begin{aligned}4\langle T(v), T(v)\rangle &= \langle T(kv + k^{-1}T(v)), kv + k^{-1}T(v)\rangle - \langle T(kv - k^{-1}T(v)), kv - k^{-1}T(v)\rangle \\ &\le |\langle T(kv + k^{-1}T(v)), kv + k^{-1}T(v)\rangle| + |\langle T(kv - k^{-1}T(v)), kv - k^{-1}T(v)\rangle| \\ &\le B\langle kv + k^{-1}T(v), kv + k^{-1}T(v)\rangle + B\langle kv - k^{-1}T(v), kv - k^{-1}T(v)\rangle \\ &\le B(2k^2\langle v, v\rangle + 2k^{-2}\langle T(v), T(v)\rangle) \\ &= 2B(k^2\langle v, v\rangle + k^{-2}\langle T(v), T(v)\rangle)\end{aligned}$$

We now let $k = \left(\frac{||T(v)||}{||v||}\right)^{1/2}$, and obtain:

$$\begin{aligned}4||T(v)||^2 = 4\langle T(v), T(v)\rangle &\le 2B(2||T(v)|| \cdot ||v||) \\ ||T(v)|| &\le B||v||.\end{aligned}$$

We conclude that $|T| = B$, as claimed. $\square$

Lemma 2. *If T is a bounded symmetric compact operator at least one of $|T|, -|T|$ is an eigenvalue for T.*

Proof. From the previous lemma we can deduce that there is a sequence of *unit* vectors v_i such that $|\langle T(v_i), v_i\rangle| \to |T|$. From this we obtain a subsequence such that $\langle T(v_i), v_i\rangle \to \lambda = \pm|T|$. Also since T is a compact operator we may assume that $T(v_i)$ converges to some vector w. We will show that $v_i \to \lambda^{-1}w$.

By the Schwartz inequality we have

$$|\langle T(v_i), v_i\rangle| \leq ||T(v_i)||\ ||v_i|| = ||T(v_i)|| \leq |T|\ ||v_i|| = |\lambda|,$$

but we know that $|\langle T(v_i), v_i\rangle| \to |\lambda$, so we conclude that $|T(v_i)| \to |\lambda|$. Consider the quantity

$$\begin{aligned} ||\lambda v_i - T(v_i)||^2 &= \langle \lambda v_i - T(v_i), \lambda v_i - T(v_i)\rangle \\ &= \lambda^2 ||v_i||^2 - 2\lambda\langle T(v_i), v_i\rangle + ||T(v_i)||^2 \\ &\to \lambda^2 - 2\lambda\lambda + \lambda^2 = 0. \end{aligned}$$

Since we know $T(v_i) \to w$, we also have that $\lambda v_i \to w$ and so we write $v_i \to v = \lambda^{-1}w$. By continuity $T(v) = \lim T(v_i) = w = \lambda v$, so we have proved that v is an eigenvector with eigenvalue λ. □

Lemma 3. *Let $T : V \to V$ be a compact operator. If $\lambda \neq 0$ is an eigenvalue of T we define $V_\lambda = \{v \in V \mid T(v) = \lambda v\}$. For any given $r > 0$, then $W = \operatorname{span}\{V_\lambda \mid \lambda \geq r\}$ is finite dimensional.*

Proof. Suppose W is not finite dimensional, then there exists a countable orthonormal subset $\{e_n\}$. Since T is a compact operator and that $||e_n|| = 1$, the sequence $\{T(e_1), T(e_2), \dots\}$ has a convergent subsequence.

However, we know that $T(e_n) = \lambda_n e_n$, with $\lambda_n \geq r$, so for $n \neq m$,

$$||T(e_n) - T(e_m)||^2 = ||\lambda_n e_n - \lambda_m e_m||^2 = ||\lambda_n e_n||^2 + ||\lambda_m e_m||^2 = \lambda_n^2 + \lambda_m^2 \geq 2r^2,$$

which contradicts the previous statement. □

From this result we conclude that every maximal orthonormal set of non-zero eigenvectors is countable. We arrange such set in a sequence (e_n) with the extra requirement that if $i < j$ and λ_i and λ_j are the eigenvalues for e_i, e_j, then $\lambda_i \geq \lambda_j$. Also from now on, we will assume $T : V \to V$ is a symmetric compact bounded operator.

Lemma 4. *Every eigenvector with a non-zero eigenvalue is a finite linear combination of vectors in (e_n).*

Proof. Let $\lambda \neq 0$ and v be said eigenvalue and eigenvector. We know that there are only finitely many n's such that e_n has characteristic value λ; we denote this set of values by S_λ. Let

$$w = v - \sum_{n \in S_\lambda} \langle v, e_n\rangle\ e_n,$$

First, note that as a linear combination of eigenvectors corresponding to λ, w is one itself; if we can show that $w = 0$, the proof will be complete. For $i \in S_\lambda$,

$$\langle w, e_i\rangle = \langle v, e_i\rangle - \sum_{n \in S_\lambda} \langle v, e_n\rangle\langle e_n, e_i\rangle = \langle v, e_i\rangle - \langle v, e_i\rangle = 0.$$

On the other hand, if $i \notin S_\lambda$ then we have $T(e_i) = \mu e_i$ with $\lambda \neq \mu$:

$$\lambda\langle w, e_i\rangle = \langle \lambda w, e_i\rangle = \langle T(w), e_i\rangle = \langle w, T(e_i)\rangle = \langle w, \mu e_i\rangle = \mu\langle w, e_i\rangle.$$

From this we conclude that $(\mu - \lambda)\langle w, e_i\rangle = 0$ and so $\langle w, e_i\rangle = 0$. Now, w must be zero, for otherwise this will contradict the maximality of (e_n).

□

0.2 Compact Groups

For the remaining part of the paper we will let G be a compact group and $V = \text{Map}\ (G, \mathbb{R})$. For consistency we will use $f, g \in V$ and $x, y, z \in G$.

Definition Let $\phi \in V$ we define $T_\phi : V \to V$ as

$$T_\phi(f)(x) = \int_G \phi(xy^{-1})f(y)dy.$$

Lemma 5. *T_ϕ is well-defined, linear and continuous. Also if $\phi(x) = \phi(x^{-1})$ for all $x \in G$, then T_ϕ is symmetric.*

Proof. Well-definedness, linearity and continuity were proved in the homework.

Suppose $\phi(x) = \phi(x^{-1})$ for all $x \in G$, then

$$\begin{aligned}
\langle T_\phi(f), g\rangle &= \int_G T_\phi(f)(x)g(x)dx = \int_G\int_G \phi(xy^{-1})f(y)dy\ g(x)dx \\
&= \int_G\int_G \phi(xy^{-1})f(y)g(x)\ dy\ dx \\
\langle f, T_\phi(g)\rangle &= \int_G f(y)T_\phi(g)(y)\ dy = \int_G f(y)\int_G \phi(yx^-1)g(x)\ dx\ dy \\
&= \int_G\int_G \phi((xy^{-1})^{-1})f(y)g(x)\ dx\ dy
\end{aligned}$$

But we know that $\phi((xy^{-1})^{-1}) = \phi(xy^{-1})$ and, if $h(x, y)$ is continuous then $\int_G\int_G h(x, y)dxdy = \int_G\int_G h(x, y)dydx$, so $\langle T_\phi(f), g\rangle = \langle f, T_\phi(g)\rangle$, and T_ϕ is symmetric. □

Definition We define the following three norms in V:

- $||f||_1 = \int_G |f(x)|dx$,
- $||f||_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_G f(x)f(x)dx}$,
- $||f||_\infty = \sup_{x\in G} |f(x)|$.

We showed in the exercises that the following inequalities hold.

$$||f||_1 \leq ||f||_2 \leq ||f||_\infty,$$

$$||T_\phi(f)||_\infty \leq ||\phi||_\infty ||f||_1.$$

Next, we recall the following (weaker version) of a result that we will need.

Theorem 1. (Arzelà-Ascoli theorem) *Let G be a compact topological group. If Δ is a uniformly equicontinuous and uniformly bounded family of functions $G \to \mathbb{R}$, then every subsequence of functions $f_i \in \Delta$ contains a uniformly convergent subsequence.*

Lemma 6. *T_ϕ is a bounded and compact operator with respect to $||\cdot||_2$.*

Proof. We proved in the homework that $||T_\phi(f)||_2 \leq ||\phi||_\infty ||f||_2$, so T_ϕ is bounded.

Let $\{f_1, f_2, \dots\}$ be a sequence of functions such that $||f_i||_2 \leq 1, \forall i$. Let Δ be the family of functions $\{T_\phi(f_1), T_\phi(f_2), \dots\}$. Then, for any $x \in G, i \in n$,

$$|T_\phi(f_i)(x)| \leq ||T_\phi(f)||_\infty \leq ||\phi||_\infty ||f_i||_1 \leq ||\phi||_\infty.$$

So Δ is uniformly bounded. On the other hand, ϕ is continuous and G compact so ϕ is uniformly continuous. Let $\epsilon > 0$, then let U be a neighborhood of the identity such that if $k \in U$ then $|\phi(kg) - \phi(g)| < \epsilon$. For any i, we have that

$$|T_\phi(f_i)(kg) - T_\phi(f_i)(g)| =$$

$$\left| \int_G \left(\phi(kgh^{-1}) - \phi(gh^{-1})\right) f_i(h)\ dh \right| \leq \int_G \left|\phi(kgh^{-1}) - \phi(gh^{-1})\right| |f_i(h)|\ dh$$

$$\leq \int_g \epsilon |f_i(h)|\ dh \leq \epsilon |f_i|_1 \leq \epsilon.$$

Now we have shown that Δ is uniformly continuous and uniformly bounded, by the Arzelà-Ascoli theorem, the sequence $\{T_\phi(f_1), T_\phi(f_2), \dots\}$ contains a convergent subsequence with respect to $||\cdot||_\infty$ and therefore also with respect to $||\cdot||_2$. We conclude that T_ϕ is a compact operator. □

We now recall the result known as the Bessel Inequality; if $\{e_n \mid n \in B\}$ is a set of orthonormal vectors in V, then for any $f \in V$

$$\sum_{n \in B} \langle f, e_n \rangle^2 \leq (||f||_2)^2.$$

The proof is as follows, consider the following non-negative quantity

$$\begin{aligned}
(||f - \sum_{n\in B} \langle f, e_n\rangle e_n||_2)^2 &= \langle f - \sum_{n\in B} \langle f, e_n\rangle e_n\ ,\ f - \sum_{n\in B}\langle f, e_n\rangle e_n\rangle \\
&= \langle f, f\rangle - 2\sum_{n\in B}\langle f, e_n\rangle\langle f, e_n\rangle + \sum_{n\in B}\sum_{m\in B}\langle f, e_n\rangle\langle f, e_m\rangle\langle e_n, e_m\rangle \\
&= (||f||_2)^2 - 2\sum_{n\in B}\langle f, e_n\rangle^2 + \sum_{n\in B}\langle f, e_n\rangle^2 \\
&= (||f||_2)^2 - \sum_{n\in B}\langle f, e_n\rangle^2 \geq 0
\end{aligned}$$

Which concludes the proof.

For the following theorem we revert to using T for the compact operator, since this is a more general result.

Theorem 2. *For any $f \in V$ we let $f_p = T(\sum_{n \le p} \langle f, e_n \rangle e_n)$. The sequence $||T(f) - f_p||_\infty$ converges to 0.*

Proof. In homework problem 11.3 we showed the first of the following inequalities;

$$||f_q - f_p||_\infty = ||T\left(\sum_{p<n\le q} \langle f, e_n \rangle e_n\right)||_\infty \le |T| \left|\left|\sum_{p<n\le q} \langle f, e_n \rangle e_n\right|\right|_2 = \left(\sum_{p<n\le p} \langle f, e_n \rangle^2\right)^{1/2}.$$

However, $\left(\sum_{p<n} \langle f, e_n \rangle^2\right)^{1/2}$ is bounded by $||f||_2$ (this follows from the Bessel inequality), and so the quantity on the right, approaches 0 as p and q become large. Hence the sequence (f_p) is Cauchy and converges to an element g of V. If we can show that $g = T(f)$ the proof will be complete. Note, however, that we have shown that we can make the quantity $||g - f_p||_2 \le ||g - f_p||_\infty$ as small as we want.

We define $T_p : V \to V$ as

$$T_p(u) = T(u) - \sum_{n \le p} \langle e_n, u \rangle T(e_n).$$

Note that $T_p(f) = T(f) - \sum_{n \le p} \langle f, e_n \rangle T(e_n) = T(f) - f_p$. So now, all we have to show is that $|T_p|$ converges to 0 as p becomes large. It is clear that T_p is bounded and compact because so is T. It is also symmetric:

$$\begin{aligned}
\langle T_p(f), g \rangle &= \langle T(f) - \sum_{n \le p} \langle f, e_n \rangle T(e_n), g \rangle \\
&= \langle f, T(g) \rangle - \sum_{n \le p} \langle f, e_n \rangle \langle T(e_n), g \rangle \\
&= \langle f, T(g) \rangle - \sum_{n \le p} \lambda_n \langle f, e_n \rangle \langle g, e_n \rangle
\end{aligned}$$

$$\begin{aligned}
\langle f, T_p(g) \rangle &= \langle f, T(g) - \sum_{n \le p} \langle g, e_n \rangle T(e_n) \rangle \\
&= \langle f, T(g) \rangle - \sum_{n \le p} \langle g, e_n \rangle \langle T(e_n), f \rangle \\
&= \langle f, T(g) \rangle - \sum_{n \le p} \lambda_n \langle g, e_n \rangle \langle f, e_n \rangle.
\end{aligned}$$

By lemma 3, there must exists some $u_p \in V$ such that $T(u_p) = \lambda_p u_p$, and $\lambda_p = \pm |T_p|$. If $\lambda_p = 0$ we are finished, otherwise, we note that for $m \le p$;

$$\begin{aligned}
\langle u_p, e_n \rangle &= 1/\lambda \ \langle T_p(u_p), e_m \rangle \\
&= 1/\lambda \ \langle u_p, T_p(e_m) \rangle \\
&= 1/\lambda \ \langle u_p, T(e_m) - \sum_{n \le p} \langle e_m, e_n \rangle T(e_n) \rangle \\
&= 1/\lambda \ \langle u_p, T(e_m) - T(e_m) \rangle = 0.
\end{aligned}$$

This means that $T_p(u_p) = T(u_p) = \lambda_p u_p$. We have shown that u_p is an eigenvalue for T and so by lemma 3, we can write $u_p = \sum_{n\in S_{\lambda_p}} \langle u_p, e_n\rangle e_n$. But then;

$$\begin{aligned}
0 \neq \lambda_p u_p = T(u_p) &= T(u_p) - \sum_{n\leq p} \langle u_p, e_n\rangle T(e_n) \\
&= T\left(\sum_{n\in S_{\lambda_p}} \langle u_p, e_n\rangle e_n\right) - \sum_{n\leq p} \langle u_p, e_n\rangle T(e_n) \\
&= \sum_{n\in S_{\lambda_p}} \langle u_p, e_n\rangle T(e_n) - \sum_{n\leq p} \langle u_p, e_n\rangle T(e_n)
\end{aligned}$$

This means that there is some $n \in S_{\lambda_p}$ that is greater than p.

In summary, we have shown that if we choose a p, we obtain u_p, λ_p and subsequently some e_n, with $T_p(e_n) = \lambda_p e_n$ such that $n > p$.

Now we let $\epsilon > 0$, and consider all the e_n's such that $|\lambda_n| > \epsilon$, (which we know to be finitely many from lemma 3); we choose p_0 greater than all these n's, in other words, if $n \geq p$ then $|\lambda_n| \leq \epsilon$. By what we proved above we obtain u_{p_0}, λ_{p_0} and e_n such that $T_{p_0}(e_n) = \lambda_{p_0} e_n$ and $n > p$, which implies $|\lambda_n| \leq \epsilon$.

So we have shown that $|T_{p_0}| \leq \epsilon$. We use this to show that we can make

$$||T(f) - f_p||_2 \leq ||T_p(f)||_2 \leq |T|\ ||f||_1$$

as small as we want. Therefore, since

$$||g - T(f)||_2 = ||g - f_p||_2 + ||f_p - T(f)||_2,$$

we have that $||g - T(f)||_2 = 0$ and the $g = T(f)$.

□

Lemma 7. *The $\lambda-$eigenspace*

$$V(\lambda) = \{f \in V | T\phi(f) = \lambda f\}$$

is invariant under R_z for all $z \in G$.

Proof. Lef f be such that $T_\phi(f) = \lambda f$. Then

$$T_\phi(R_z(f))(x) = \int_G \phi(xy^{-1}) R_z(f)(y) dy = \int_G \phi(xy^{-1}) f(yz) dy.$$

We make the change of variables $y \to yz^{-1}$ and obtain

$$\int_G \phi(xzy^{-1}) f(y) dy = T_\phi(f)(xz) = R_z(T_\phi(f))(x) = \lambda R_z(f)(x).$$

So $R_z(f) \in V(\lambda)$.

□

Lemma 8. *Let G be a compact group, and U a neighborhood of the identity. Then we can find a function ϕ supported in U, such that $\phi(x) = \phi(x^{-1})$, $\forall x \in G$, and $\int_G \phi(x)\ dx = 1$.*

Proof. We first define V an open neighborhood of the identity such that $V = V^{-1} \subset U$. G is compact and Hausdorff, also $\{e\}$ and $G \backslash V$ are closed and so, by Urysohn's Lemma, we obtain a function $\phi'' : G \to \mathbb{R}$ such that

$$\phi''(e) = 1, \phi''(y) = 0, \ \forall y \notin V.$$

We next define a function $\phi' : G \to \mathbb{R}$ as $\phi'(x) = \phi''(x) + \phi''(x^{-1})$. It is clear that $\phi'(x) = \phi'(-x)$. Finally, we define

$$\phi(x) = \frac{\phi'(x)}{\int_G \phi'(z) \ dz}.$$

Then, ϕ is the desired function:

- For $y \notin U$ we know that $y, y^{-1} \notin V$, and so we have that

$$\phi(y) = \frac{\phi''(y) + \phi''(y^{-1})}{\int_G \phi'(z) \ dz} = 0,$$

  so ϕ is supported in U.

- Finally,

$$\int_G \phi(x) \ dx = \int_G \frac{\phi'(x)}{\int_G \phi'(z) \ dz} \ dx = 1.$$

□

The last thing we will need in order to prove the Peter-Weyl theorem is the following fact, which was proved in homework 9.3.

If the vector subspace of V spanned by $\{R_x(f) \mid x \in G\}$ is finite dimensional, then f is a matrix representation.

This is completes all the prerequisites we need to prove our main result.

Theorem 3. (Peter-Weyl Theorem) *Let G be a compact group, then the matrix coefficients of G are dense in V.*

Proof. Let $f \in V$ and $\epsilon > 0$. We will show that there exists a matrix coefficient f' such that $||f - f'||_\infty < \epsilon$.

Since G is compact, f is uniformly continuous, and so we can find U, an open symmetric neighborhood of the identity such that, if $x \in U$ such that

$$||L_x(f) - f||_\infty < \epsilon/2.$$

(Note, this was proved in homework 7.3.) By lemma 8, we can find $\phi \in V$, supported in U and such that

$$\phi(x) = \phi(x^{-1}), \text{ and } \int_G \phi(x) \ dx = 1.$$

We thus obtain $T_\phi : V \to V$, which is symmetric and compact. We claim that $||T_\phi(f) - f||_\infty < \epsilon/2$. For any $x \in G$,

$$
\begin{aligned}
|T_\phi(f)(x) - f(x)| &= \left| \int_G (\phi(xy^{-1})f(y)\ dy - \int_G \phi(y)f(x)\ dy \right| \\
&\le \int_G \left| \phi(y)f(y^{-1}x) - \phi(y)f(x) \right|\ dy \\
&= \int_G \phi(y) \left| f(y^{-1}x) - f(x) \right|\ dy \\
&= \int_U \phi(y) \left| f(y^{-1}x) - f(x) \right|\ dy \\
&\le \int_U \phi(y) ||L_{y^{-1}}(f) - f||_\infty\ dy \\
&\le \int_U \phi(y)(\epsilon/2)\ dy = \epsilon/2.
\end{aligned}
$$

By theorem 1, we can choose p so that

$$||T_\phi(f) - f_p||_\infty < \frac{\epsilon}{2}.$$

Recall that

$$f_p = T_\phi \left(\sum_{n \le p} \langle f, e_n \rangle e_n \right) = \sum_{n \le p} \langle f, e_n \rangle \lambda_n e_n,$$

and notice that it is contained in a finite dimensional vector space, which, by lemma 7 is closed under right translation, we conclude that f_p is a matrix coefficient

Finally,

$$||f - f'||_\infty = ||f - T_\phi(f) + T_\phi(f) - f_p||_\infty \le ||f - T_\phi(f)||_\infty + ||T_\phi(f) - f_p||_\infty \le \epsilon.$$

□

We will now show some applications of the theorem as the following two corollaries.

Corollary 1. *Suppose G is a compact group, H a closed subgroup and U a neighborhood of e. Then, there exists a finite dimensional linear G-space V and $v \in V$ such that*

$$H \subset G_v \subset UH.$$

Proof. G/H is compact and Hausdorff, so by Urysohn's Lemma, there exists a continuous function $\tilde{f} : G/H \to I = [0, 1]$ such that

- $\tilde{f}(eH) = 0$,
- $\tilde{f}(y) = 1,\ \forall y \notin \pi(UH)$.

We define $f' = \tilde{f} \circ \pi : G \to I$ and consider the following statements:

1. $f'(h) = 0,\ \forall h \in H$,
2. $f'(x) = 1,\ \forall x \notin UH$,
3. $R_h f = f,\ \forall h \in H$

Statements 1 and 2 are trivial, and 3 follows because $\pi(xh) = \pi(x)$.

By the Peter-Weyl theorem we find $f'' : G \to \mathbb{R}$, a *matrix coefficient* such that $|(f'' - f')(x)| < \epsilon$, $\forall x \in G$. Unfortunately f'' does not necessarily satisfy the given conditions, but it does have some useful properties.

1. For all $h \in H, f''(h) < f'(h) + \epsilon = \epsilon$. For $x \notin UH$
2. For all $x \notin UH, f''(x) > f'(x) - \epsilon = 1 - \epsilon$.
3. Unfortunately, there is nothing like property 3 that applies to f'', so we must define yet another function.

We let $f : G \to I$ as $f(x) = \displaystyle\int_H f''(xh')\ dh'$. Since f'' is a matrix coefficient, so is f.

1. For $h, h' \in H$, we show that $f(h) = \int_H f''(hh')\ dh' < \int_H \epsilon = \epsilon$.
2. For $x \notin UH$, we show that $f(x) = \int_H f''(xh)\ dh' > \int_H 1 - \epsilon = 1 - \epsilon$.
3. For $x \in G, h \in H$ we show that $f(xh) = \int_H f''(xhh')\ dh' = \int_H f''(xh')\ dh = f(x)$.

Consider the action $R : G \times V_f \to V_f$, we will show that V_f and $f \in V_f$ satisfy the conclusion of the theorem:

- Firstly, since f is a matrix coefficient, we know V_f is finite dimensional.
- For any $h \in H$, we showed $R(h, f) = R_h f = f$, so $H \subset G_f$.
- For $x \notin UH$ consider $R_x f(e) = f(x) > 1 - \epsilon$, but $f(e) < \epsilon$, since $e \in H$. If we also require that $\epsilon < 1/2$, this shows that $x \notin G_f$, and so $G_f \subset UH$.

□

Corollary 2. *Suppose G is a compact group, U is an open neighborhood of the identity. Then there exists a linear representation $\phi : G \to O(n)$, for some $n \in \mathbb{N}$ such that* $\mathrm{Ker}(\phi) \subset U$

Proof. By applying corollary 1 to $H = \{e\}$ we obtain $\psi : G \to (V)$ such that for some $v \in V$, $\mathrm{Ker}(\psi) \subset G_v \subset U$.

We showed in lectures (pdf 1 p.38) that ψ is equivalent to an orthogonal representation $\phi : G \to O(n)$, which satisfies all the conditions of the corollary. □

For the next corollary we will need the following definition. We say that a compact group G *has no small subgroups*, is there exists a neighborhood U of the identity such that no non-trivial subgroups are contained in U.

Corollary 3. *Let G be a compact group that has no small subgroups. Then G is homeomorphic so a subgroup of $O(n)$ for some n.*

Proof. We let U be the neighborhood of identity defined in the previous definition, and set $H = e$. Then, by the previous corollary we find a representation $f : G \to O(n)$ such that $\mathrm{Ker}(f) \subset U$, so by our choice of U we deduce that $\mathrm{Ker}(f)$ is trivial, and f is an injection.

□