

EXAM

Mathematical Modelling

13 August 2009

Question 1: The life cycle of many butterfly species is linked to that of a parasitoid, which is another insect species (usually a wasp) that lies its eggs inside the larvae of the butterfly so that the larvae do not develop into a new butterfly but into one or more individuals of the parasitoid instead. Suppose that

1. each adult butterfly produces new larvae at a constant rate,
2. parasitoid individuals search randomly for butterfly larvae,
3. whenever a larva is found by a parasitoid, then the parasitoid deposits a single egg inside the larva, independently of the number of eggs that may already be inside,
4. a larva containing zero eggs of the parasitoid develops into an adult butterfly after an exponentially distributed amount of time,
5. a larva containing a single egg develops into an adult parasitoid after an exponentially distributed amount of time,
6. a larva with two or more eggs inside does not develop into anything and eventually simply dies,
7. larvae, butterflies and parasitoids each have their own death rate.

Give a model of the above situation by first **(a)** distinguishing all the necessary individual states, then **(b)** representing the above processes as unimolecular or bimolecular reactions, and finally **(c)** formulating the corresponding differential equations for changes in the population densities.

Question 2: Consider the system

$$\frac{dX}{dt} = \alpha - \varepsilon X - \beta XY$$

$$\frac{dY}{dt} = \gamma\beta XY - \delta Y$$

for $\alpha, \beta, \gamma, \delta > 0$ and $X, Y \geq 0$. **(a)** What kind of processes may be described by these equations? Give interpretations of the parameters and variables. **(b)** Perform a phase plane analysis of the system, and determine the stability of all equilibria. If the phase plane method is inconclusive as far as stability is concerned, then give a local stability analysis. **(c)** Show that limit cycles are not possible. *Hint:* use the Dulac function $u(X, Y) = 1/Y$.

Question 3: Consider a situation where individuals move randomly, but each time two individuals meet, they stop and shake hands (or interact in some other way) for an exponentially distributed period of time. This can be modeled by the following system:

$$\frac{\partial S}{\partial t} = -\alpha S^2 + 2\beta P + D \frac{\partial^2 S}{\partial x^2}$$

$$\frac{\partial P}{\partial t} = \frac{1}{2}\alpha S^2 + \beta P$$

where $\alpha, \beta, D > 0$ and where S is the population of single individuals and P that of pairs. **(a)** Rewrite this system for the total population density $N := S + 2P$ and the population density of singles S . **(b)** Assuming that diffusion is slow compared to the other processes, the system becomes a slow-fast system with N as the slow variable and S as the fast variable. Calculate the quasi-equilibrium for S , show that it is stable. **(c)** Use the quasi-equilibrium to get a single closed equation for the slow variable N . **(d)** Calculate the equilibrium for N assuming a one dimensional domain $[0, 1]$ with boundary conditions $N(t, 0) = N_0 > 0$ and $N(t, 1) = 0$ for all t .

Question 4: Suggest a set of **(a)** partial differential equations and **(b)** boundary conditions for the following system in a one-dimensional domain with reflecting boundary conditions at zero and one: Bacterium cells (B) produce a chemical substance (C). The movement of the bacteria is random, and the chemical spreads by diffusion. There are also amoeba cells (A) that feed on the bacteria. The amoebae show positive taxis towards concentrations of the chemical, but their movement also contains a random undirected component.

Success!