31-1-2012

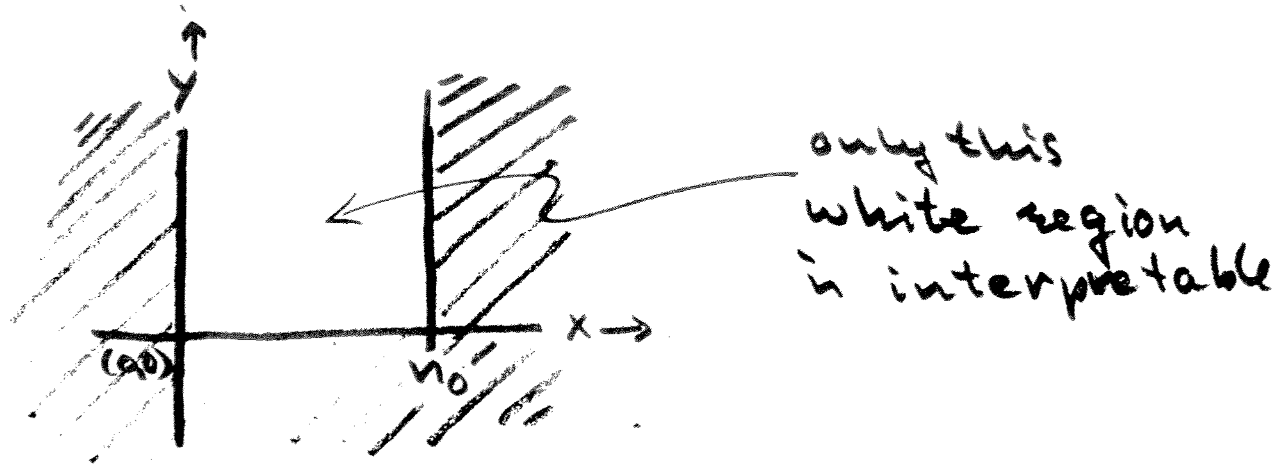
Previous lecture we derived the population model

$$\begin{cases} \dfrac{dx}{dt} = +\beta(n_0 - x)y - \mu x \\ \dfrac{dy}{dt} = -\beta(n_0 - x)y + \alpha x - \nu y. \end{cases}$$

(see prev. lecture for interpretation).

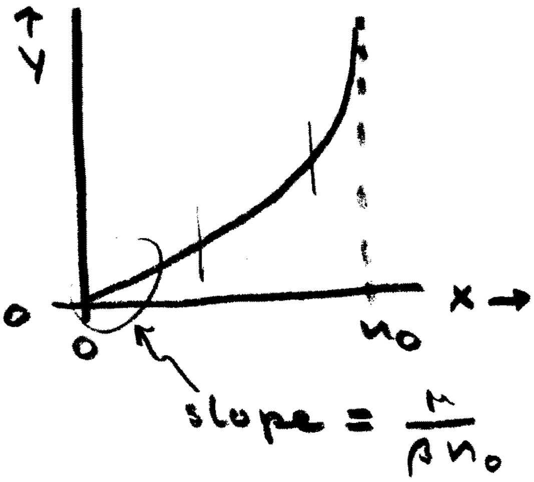
Here we analyze the model using the <u>phase-plane method</u>.

The phase-plane (or population state space) is the set of all feasible values of (x, y); here $\{(x,y) : 0 \leq x \leq n_0 \;\&\; 0 \leq y\}$.

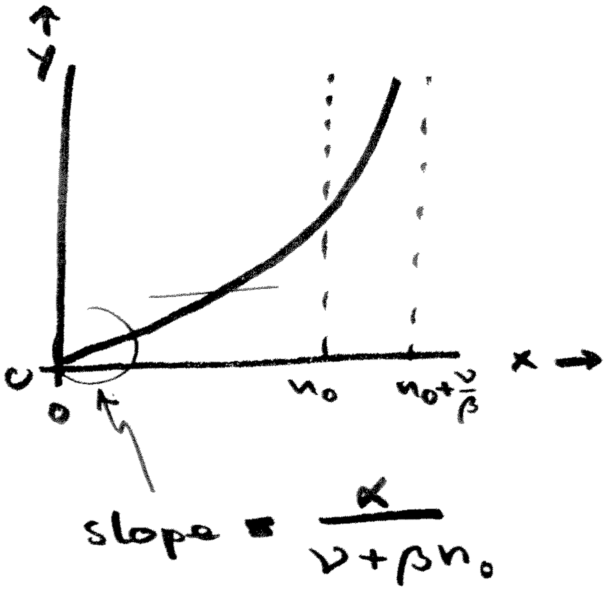


The <u>zero-cline for x</u> is the set $\{(x,y) : \frac{dx}{dt} = 0\}$ and is crossed vertically from above or from below:

$$\boxed{\frac{dx}{dt} = 0 \iff y = \frac{\mu}{\beta} \cdot \frac{x}{n_0 - x}}$$



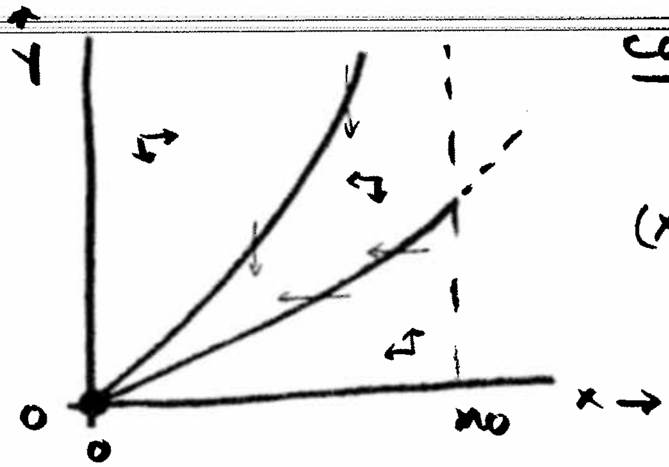
The zero-cline for y is the set $\{(x,y) : \frac{dy}{dt} = 0\}$ and is crossed horizontally from the left or from the right.



$$\boxed{\frac{dy}{dt} = 0 \iff y = \frac{\alpha}{\beta} \cdot \frac{x}{\frac{\nu}{\beta} + n_0 - x}}$$

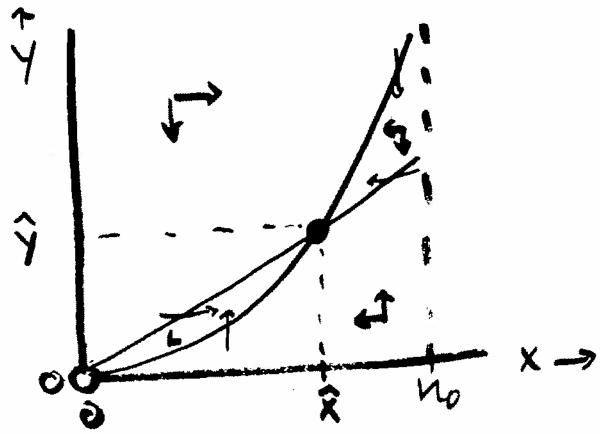
Combining the zero-clines in one picture gives

case $\boxed{\dfrac{\alpha}{\nu+\beta n_0} \leq \dfrac{\mu}{\beta n_0}}$



$(x,y) = (0,0)$ globally attracting equilibrium.

case: $\boxed{\dfrac{\alpha}{\nu+\beta n_0} > \dfrac{\mu}{\beta n_0}}$



$(x,y) = (0,0)$ unstable equilibrium

$(x,y) = (\hat{x},\hat{y})$ globally attracting equilibrium

$$\boxed{\begin{aligned} \hat{x} &= n_0 - \frac{\mu\nu}{\beta(\alpha-\mu)} \\ \hat{y} &= \frac{n_0(\alpha-\mu)}{\nu} - \frac{\mu}{\beta} \end{aligned}}$$

an equilibrium is any point (x,y) where $\frac{dx}{dt} = 0$ <u>and</u> $\frac{dy}{dt} = 0$ (i.e. any intersection of the zero-clines)

What is the expected life time of a newborn at the positive equilibrium?

Rewrite the reaction network as

$$\begin{tikzcd}
\boxed{\text{entry}} \arrow[r] & Y \arrow[r, "\beta(n_0-\hat{x})"] \arrow[d, "\nu"] & X \arrow[d, "\mu"] \\
 & \boxed{\text{exit}} & \boxed{\text{exit}}
\end{tikzcd}$$

(birth) (death) (death)

The transitions at equilibrium are all Poisson processes with a <u>constant</u> rate.

Expected lifetime of newborn:

$$T = \underbrace{\frac{1}{\nu+\beta(n_0-\hat{x})}}_{\text{expected time spent in state } Y} + \underbrace{\frac{\beta(n_0-\hat{x})}{\nu+\beta(n_0-\hat{x})}}_{\text{probability transition } Y \to X} \cdot \underbrace{\frac{1}{\mu}}_{\text{expected time spent in } X.}$$

What is the expected lifetime number of offspring for a newborn at equilibrium?

$$\begin{tikzcd} \boxed{\text{entry}} \arrow[r] & Y \arrow[r, "\beta(n_0-\hat{x})"] \arrow[d, "\nu"] & X \arrow[r, squiggly, "\alpha"] \arrow[d, "\mu"] & \text{offspring} \\ & \boxed{\text{exit}} & \boxed{\text{exit}} & \end{tikzcd}$$

Expected number of offspring:

$$E = \boxed{\frac{\beta(n_0-\hat{x})}{\nu+\beta(n_0-\hat{x})}} \cdot \boxed{\frac{1}{\mu}} \cdot \boxed{\alpha}$$

probability of reaching state X.

expected time in state X.

expected number of offspring per unit of time while in X.

The present modelling approach readily leads to large systems of differential equations that are ~~often difficult to analyze.~~

Example

i-states: N prey individual.

X searching predator.

Y attacking pred-prey pair.

Z resting predator.

i-level processes

① $X + N \xrightarrow{\alpha} Y$ (find prey and attack.)

② $Y \xrightarrow{\beta} Z$ (prey capture)

③ $Y \xrightarrow{\gamma} X + N$ (prey escape)

④ $Z \xrightarrow{\delta} X$ (resume search)

continued →

$\Rightarrow$ p-level equations:

$$
\left\{
\begin{array}{rcl|c|c|c}
 & & ① & ② & ③ & ④ \\
\dfrac{dn}{dt} & = & -\alpha x n & & +\gamma y & \\
\dfrac{dx}{dt} & = & -\alpha x n & & +\gamma y & +\delta z \\
\dfrac{dy}{dt} & = & +\alpha x n & -\beta y & -\gamma y & \\
\dfrac{dz}{dt} & = & & +\beta y & & -\delta z
\end{array}
\right.
$$

4D system

Conservation rule:

Total predator density
$x + y + z = x_0$ is constant.
Use this to eliminate z:

$$
(*) \left\{
\begin{aligned}
\frac{dn}{dt} &= -\alpha x n + \gamma y \\
\frac{dx}{dt} &= -\alpha x n + \gamma y + \delta (x_0 - x - y). \\
\frac{dy}{dt} &= +\alpha x n - \beta y - \gamma y
\end{aligned}
\right.
$$

This is still a 3D system:
awkward to analyze.

→ Remedy: Time-scale separation.

- ~~Suppose that the involved processes of prey capture ② and prey escape ③ are fast~~ compared to the other processes.
- ~~Also~~ Suppose that the prey population is large compared to that of the predator.

(These are more or less realistic assumptions, by the way)

Then we can split the 3D system into two smaller systems of fast and slow variables and study them separately.

How this is done, we'll see in the next lecture.