Previous lecture

i-level:		p-level.
Individual behavior	$\Longrightarrow$	Population behavior
$\uparrow$		$\uparrow$
modeled		derived.

Basic processes on i-level

- Interactions & reactions
- Growth & development
- movement in space.

Interactions & reactions

- Monomolecular reaction (MR)
- Bimolecular reaction (BR)
- Reaction networks.

Monomolecular reaction

|i-level| A : single individual

$$A \xrightarrow{\lambda} \text{products}$$

(Poisson-process with rate λ)

|p-level| a : population density

$$\frac{da}{dt} = -\lambda a$$

(Law of the large numbers)

Biological examples

Cell division
Asexual reproduction
Maturation
Death
Recovery from illness

... may all be modelled by monomolecular reactions.

Today:

Bimolecular reaction

(*) $A + B \longrightarrow$ products.

Two particles meet and react.
For a given A particle, the number of meetings with any B particle per unit of time is proportional to the concentration b of B particles.

$\Rightarrow$ (*) can also be represented by the reaction

$$A \xrightarrow{\lambda b} \text{product}$$

(Poisson process with rate λb)

$$\Rightarrow \quad \frac{da}{dt} = -(\lambda b)\,a = -\lambda ab$$

Likewise:

$$\frac{db}{dt} = -(\lambda a)\,b = -\lambda ab$$

Principle of mass-action

In a bimolecular reaction,

the probability per unit of time that a *given* particle undergoes a reaction is proportional to the concentration of the kind of particles it is reacting with.

Bimolecular reaction

i-level A, B individuals of different kind

$$A + B \xrightarrow{\lambda} \text{products}$$

(assuming mass action)

p-level a, b population densities

$$\begin{cases} \dfrac{da}{dt} = -\lambda ab \\[2ex] \dfrac{db}{dt} = -\lambda ab \end{cases}$$

(Law of large numbers).

[1] Example

| i-level | P predator individual

N prey individual

$$N + P \xrightarrow{\lambda} P$$

(prey capture)

| p-level | p, n pop. dens

$$\begin{cases} \frac{dp}{dt} = 0 \\ \frac{dn}{dt} = -\lambda n p . \end{cases}$$

[2nd] Example

| i-level | I: infected, S: susceptible

(*) $S + I \xrightarrow{\beta} 2I$ (transmission)

p-level i, s pop dens.

$$\begin{cases} \frac{ds}{dt} = -\beta s i \\ \frac{di}{dt} = \underbrace{-\beta s i}_{\text{left side of } (*)} + \underbrace{2\beta s i}_{\text{right side of } (*)} = +\beta s i \end{cases}$$

Bimolec. reaction between two identical particles:

$$A + A \xrightarrow{\lambda} \text{products.}$$

Write as

$$A \xrightarrow{\lambda a} \text{products.}$$

Hence

$$\frac{da}{dt} = -(\lambda a)\, a = -\lambda a^2 .$$

(and not $\frac{da}{dt} = -2\lambda a^2$)

Example.

i-level A : individual

⊛ $2A \xrightarrow{\lambda} A$

(interference competition)

p-level a : pop. dens.

$$\frac{da}{dt} = -\lambda a^2 + \frac{1}{2}\lambda a^2 = -\frac{1}{2}\lambda a^2$$

Left side of ⊛

right side of ⊛:
(A re-appears at half the rate as A disappears on the left)

Reaction networks.

i-level | Example.

X : individual.

① $X \xrightarrow{\lambda} 2X$ (asexual reproduction)

② $X \xrightarrow{\mu} \dagger$ (death)

③ $2X \xrightarrow{\nu} X$ (interference comp.)

p-level

x : pop. dens.

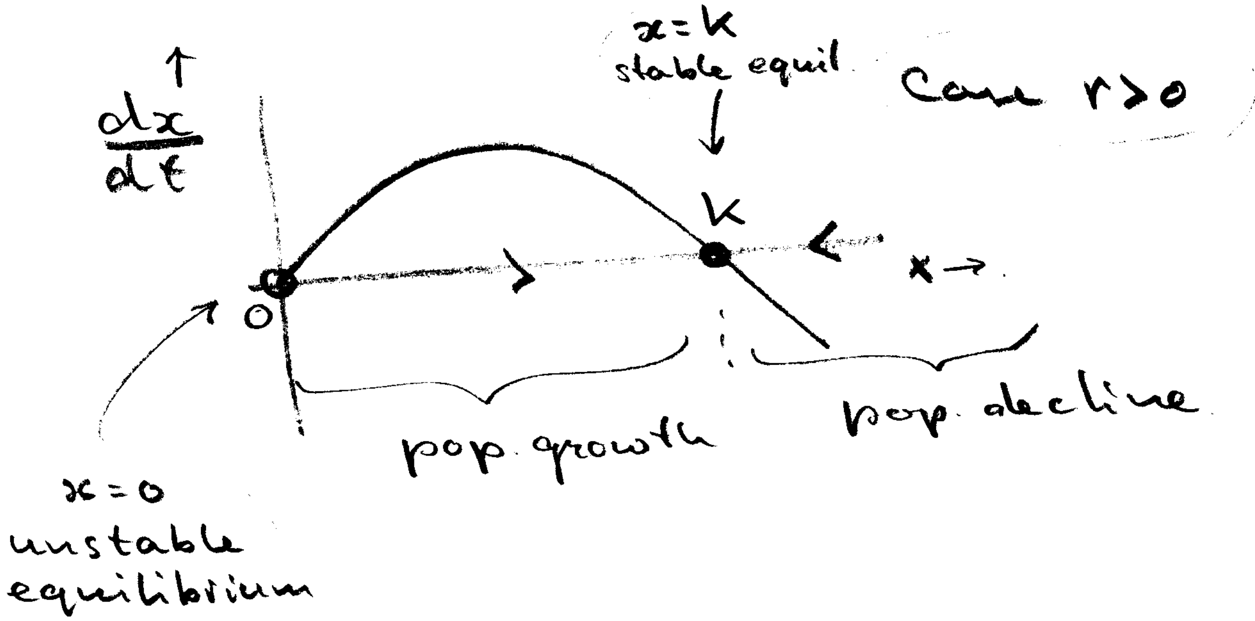
$$\frac{dx}{dt} = \overbrace{\underbrace{-\lambda x}_{\substack{\text{left}\\ \text{(reactant)}}} + \underbrace{2\lambda x}_{\substack{\text{right}\\ \text{(product)}}}}^{①} \overbrace{- \mu x}^{②} \overbrace{\underbrace{- \nu x^2}_{\substack{\text{left}\\ \text{(reactant)}}} + \underbrace{\tfrac{1}{2}\nu x^2}_{\substack{\text{righ.}\\ \text{(product)}}}}^{③}$$

$$= (\lambda - \mu)x - \tfrac{1}{2}\nu x^2$$

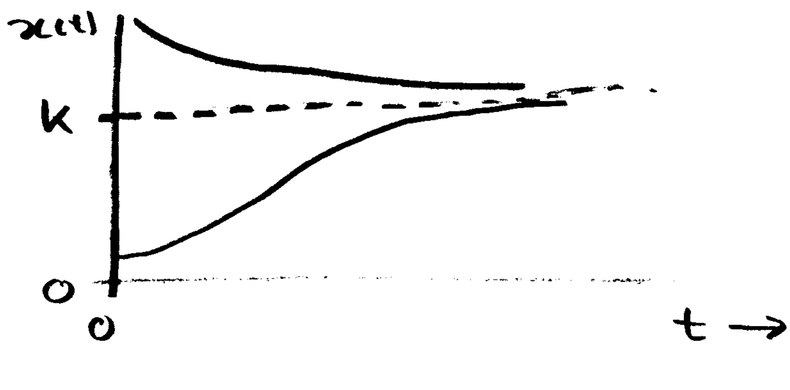
model analysis:

Rewrite as

$$\begin{cases} \dfrac{dx}{dt} = r x \left(1 - \dfrac{x}{K}\right) & \text{(logistic)} \\ r = \lambda - \mu \\ K = \dfrac{2(\lambda - \mu)}{\nu} \end{cases}$$



⇒ Qualitative behavior of x(t) for two different starting points



Another example

i-level

these are called the i-states:

N : unoccupied territory

X : occupied territory
(= territory owner)

Y : individual without territory

(1) $N + Y \xrightarrow{\beta} X$ (collonization)

(2) $X \xrightarrow{\alpha} X + Y$ (reproduction)

(3) $X \xrightarrow{\mu} N + \dagger$ (death)

(4) $Y \xrightarrow{\nu} \dagger$ (death

p-level

$$\left\{\begin{array}{lcccc} \frac{dx}{dt} = & \beta n y & & -\mu x & \\ \frac{dy}{dt} = & -\beta n y & +\alpha x & & -\nu y \\ \frac{dn}{dt} = & -\beta n y & & +\mu x & \\ & \underbrace{\qquad\qquad}_{(1)} & \underbrace{\qquad\qquad}_{(2)} & \underbrace{\qquad\qquad}_{(3)} & \underbrace{\qquad\qquad}_{(4)} \end{array}\right.$$

$\boxed{\text{analysis}}$

Notice that $\frac{d(n+x)}{dt} = 0$,

$\Rightarrow$ $n + x =: n_0$ is constant

(total density of territories, occupied or unoccupied).

Use this to eliminate n from the equations:

$$\Rightarrow \begin{cases} \frac{dx}{dt} = \beta(n_0 - x)y - \mu x \\ \frac{dy}{dt} = -\beta(n_0 - x)y + \alpha x - \nu y \end{cases}$$

Next lecture we analize this system further ...