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Discrete-time pop. models

(as an application of the modeling tools we've developed so far)

$x_n \in \mathbb{R}_+^k$: k-dim. vector of pop. densities at time $n \in \{0,1,2,\dots\}$.

Dynamics of x given by

$\boxed{x_{n+1} = \varphi(x_n)}$ (discrete dyn. syst.)

where $\varphi : \mathbb{R}_+^k \longrightarrow \mathbb{R}_+^k$ continuous.

$$x_n \longrightarrow \boxed{\begin{array}{c}\text{processes}\\ \text{within}\\ \text{season}\end{array}} \longrightarrow x_{n+1}$$

The processes within the season define the map $x_n \xmapsto{\varphi} x_{n+1}$.

$\Rightarrow$ We can <u>derive</u> the function φ from explicitly modelling the continuous time processes within the season

Example (resource - consumer model)

(Geritzakisdi, J. Theor. Biol. (2004) 228: 261–269)

Within-season dynamics

$$(1) \begin{cases} \dfrac{dR_n}{dt} = \alpha R_n f(R_n) - \beta R_n x_n & \text{(resource)} \\ \dfrac{dE_n}{dt} = \gamma \beta R_n x_n - \delta E_n & \text{(eggs of consumer)} \\ \dfrac{dx_n}{dt} = 0 & \text{(adult consumer).} \end{cases}$$

(Interpret the various terms!)

Season number $n = 0, 1, 2, \ldots$

Time within season $t \in [0,1]$.

Between-season dynamics

$$(2) \begin{cases} R_{n+1}(0) = \rho R_n(1) \\ E_{n+1}(0) = 0 \\ x_{n+1}(0) = \sigma E_n(1) \end{cases}$$

$\rho \in (0,1)$ between-season survival probability of resources.

$\sigma \in (0,1)$ between-season survival and hedging probability of eggs.

Note

- The within-season dynamics is in continuous time $t \in [0,1]$ and is based on mass-action
- The between-season dynamics is in discrete time $n \in \{0,1,2,\dots\}$ and connects the initial conditions for one season the the final conditions of the previous season

Assume

- Within season dynamics without consumers

$$\boxed{\frac{dk_n}{dt} = \alpha k_n f(k_n)}$$

with

- $f : \mathbb{R}_+ \to \mathbb{R}$ is monotonically decreasing
- $\exists k > 0 : f(k) = 0$

$$\cancel{\lim_{k \to 0} k f(k) < \infty}$$

- Also assume fast-slow within-season dynamics with $\boxed{\alpha, \beta \ll \gamma\beta, \delta}$:

$$(3)\quad \begin{cases} \dfrac{dR_n}{dt} = \alpha R_n f(R_n) - \beta R_n x_n & (\underline{\text{fast}}) \\[2mm] \dfrac{dE_n}{dt} = \gamma\beta R_n x_n - \delta E_n & (\underline{\text{slow}}) \\[2mm] \dfrac{dx_n}{dt} = 0 & (\underline{\text{constant}}) \end{cases}$$

Then the (fast) resource dyn. has a stable (quasi) equil.

$$(4)\quad \hat{R}_n := \begin{cases} f^{-1}\left(\frac{\beta}{\alpha} x_n\right) & \text{if } x_n \leq x^* \\ 0 & \text{if } x_n > x^* \end{cases}$$

where

$$(5)\quad x^* := \frac{\alpha}{\beta} \lim_{R \downarrow 0} f(R) \leq \infty$$

(If $x_n > x^*$, the resource is over-exploited.
The value x^* is found from $0 = f^{-1}\left(\frac{\beta}{\alpha} x^*\right)$)

NB. $\hat{R}_n$ is constant <u>within</u> a season because x_n is constant.

The slow dyn. then become

(6) $$\begin{cases} \dfrac{dE_n}{dt} = \gamma\beta\hat{R}_n x_n - \delta E_n \\ E_n(0) = 0 \qquad \text{(see (2) page 53)} \end{cases}$$

which is explicitly solved as

(7) $$E_n(t) = \frac{\gamma\beta}{\delta}(1 - e^{-\delta t})\hat{R}_n x_n$$

For the between-season dyn. (2) (p. 53) we thus have

(8) $$x_{n+1} = \sigma \cdot \frac{\gamma\beta}{\delta}(1 - e^{-\delta}) \cdot \hat{R}_n \cdot x_n$$

and hence, from (4) (p. 55),

(9) $$x_{n+1} = \begin{cases} \lambda x_n f^{-1}\left(\frac{\beta}{\alpha} x_n\right) & \text{if } x_n \le x^* \\ 0 & \text{if } x_n > x^* \end{cases}$$

with

(10) $$\lambda := \sigma \frac{\gamma\beta}{\delta}(1 - e^{-\delta})$$

and where x^* is given by (5) on page 55.

Note

Equation (9) gives a link between the <u>within-season resource dynamics in absence of consumers</u>

(*) $$\boxed{\frac{dR}{dt} = \alpha R f(R)}$$

to the between-season consumer dynamics, which is of the form

(**) $$x_{n+1} = \begin{cases} \lambda x_n f^{-1}\left(\frac{\beta}{\alpha} x_n\right) & \text{if } x_n \leq x^* \\ 0 & \text{otherwise} \end{cases}$$

We can use this to derive the between-season dyn. from a given type of resource dynamics.

We can also use this in the reverse direction to find the resource dynamics that will lead to a given between-season consumer dynamics.

This gives, e.g.,

(a) $\frac{dR}{dt} = \alpha R(1 - \frac{R}{K})$ (logistic)

$\Longleftrightarrow$ $x_{n+1} = \begin{cases} a x_n (1 - \frac{x_n}{b}) & \text{if } x_n \le b \\ 0 & \text{otherwise} \end{cases}$

(discrete logistic)

(b) $\frac{dR}{dt} = \alpha R^{\theta} - \alpha K^{\theta - 1} R$ (Von Bertalanffy)

$\Longleftrightarrow$ $x_{n+1} = \frac{a x_n}{(1 + b x_n)^c}$ (Hassel, 1975)

(c) $\frac{dR}{dt} = \alpha R (1 - \frac{\log R}{\log K})$ (Gompertz)

$\Longleftrightarrow$ $x_{n+1} = a x_n e^{-b x_n}$ (Ricker, 1954)

(d) $\frac{dR}{dt} = \alpha - \alpha \frac{R}{K}$ (chemostat)

$\Longleftrightarrow$ $x_{n+1} = \frac{a x_n}{1 + b x_n}$ (Beverton-Holt, 1957)