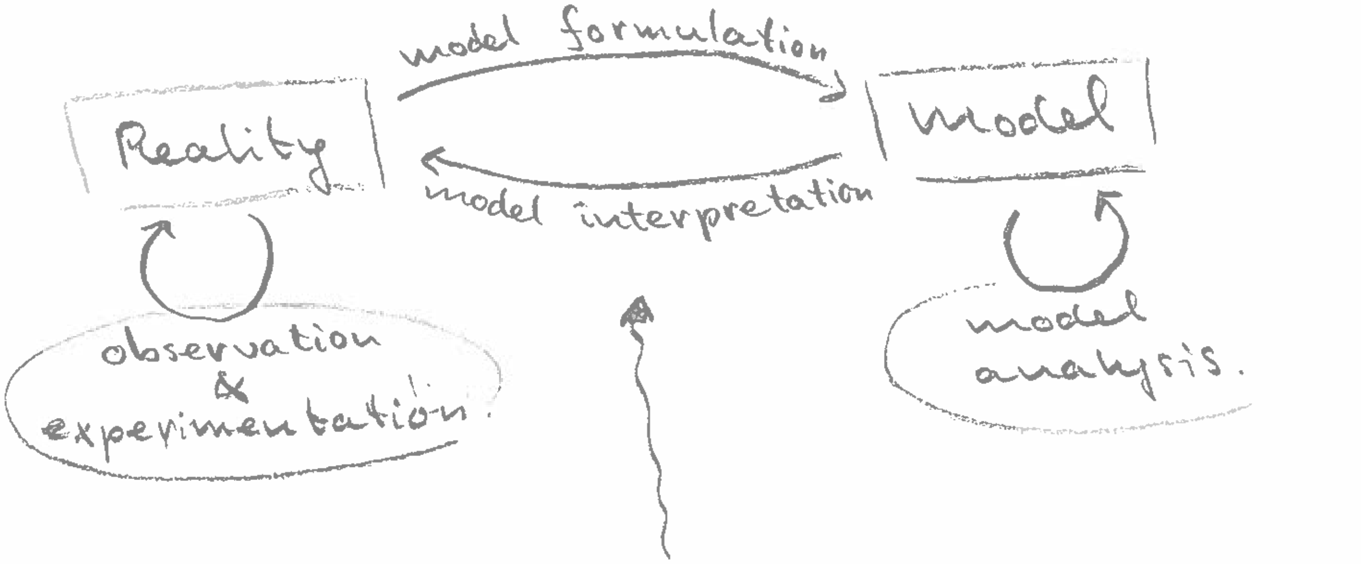
Mathematical modeling (2013)

The empirical scientist studies reality through observation and experimentation.

The mathematician studies a model of reality through mathematical analysis. In this way he learns about the model, but to what extent he learns about reality depends on how the model relates to the real thing.



① The emphasis of the course lies here: model formulation and interpretation. If this interface between the model and reality is not good, model analysis is just a formal exercise without real significance.

Modeling approach

Population = collection of individuals

Population behavior: changes in size, composition, distribution in space

Individual behavior: birth, death, growth and development, movement in space, interaction with other individuals.

② The second theme of this course is the derivation of a population model from a model of the behavior of the individuals that make up the population.

$\boxed{\text{model of individual behavior}} \xrightarrow{\text{derivation}} \boxed{\text{population model}}$

(formulated) (derived)

Some terminology

Level of the individual (i-level)	Level of the population (p-level)
i-state	p-state

i-state

- E.g., size of an individual, its age, gender, location, etc., or a combination of these.
- The i-state of an individual fully determines its behavioral options. E.g., a juvenile cannot reproduce, but an adult can.
- The i-state space is the set of all possible i-states. E.g.,

$$\Omega = \{M, F\} \times \{J, A\}$$

i-state space. male female juvenile adult

$(M, J) \in \Omega$ is the state "male & juvenile", etc.

p-state

- most generally the p-state is a positive measure on the i-state space Ω that tells us how many individuals there are in a given subset of i-states.
- If Ω is finite, the measure is fully determined by the population densities of each element of Ω. E.g., if $\Omega = \{M, F\} \times \{J, A\}$ (see left panel), then the p-state is given by the pop. densities

$$n_{(M,J)},\ n_{(F,J)},\ n_{(M,A)},\ n_{(F,A)}$$

male juveniles per unit of area etc.

(continued)

i-level	p-level
i-processes • Individuals can change their i-state through growth, movement in space, interaction with others.	• Changes in the i-states lead to changes in de p-states, described by differential equations or finite difference equations.

The formulation of a model of individual behavior amounts to specifying the i-states as well as the i-processes.

The subsequent derivation of the corresponding model for a collection of (many) individuals needs extra assumptions, such as large population size.

(Example follows)

Example

Formulation of the model of individual behavior:

This is the model.

i-states: alive, dead.

i-processes: dying as a Poisson-process with constant rate parameter $\alpha > 0$.

Convenient shorthand notation:

(x) : alive individual

(y) : dead individual

$(x) \xrightarrow{\alpha} (y)$: process of dying.

Dying as a Poisson with a constant rate $\alpha > 0$ means that an individual has a constant probability per unit of time of dying, independently of how long it has been alive.

The probability of dying in the next h time units is $\boxed{\alpha h + O(h^2)}$ (where $O(h^2)$ denote rest terms that become zero at a rate at least as fast as h^2 as $h \to 0$.)

Let $P(t)$ denote the probability that an individual that was alive at time zero is still alive at time t.

Then for small $h>0$:

$P(t+h) =$

$= P(t) \cdot \mathrm{Prob}\{\text{no death during } (t, t+h]\} =$

$= P(t) \cdot (1 - \mathrm{Prob}\{\text{death during } (t, t+h]\} =$

$= P(t)\,(1 - \alpha h + O(h^2))$

Rearranging the equation gives:

$$\underbrace{\frac{P(t+h)-P(t)}{h}}_{\downarrow h \to 0} = \underbrace{-\alpha P(t) + O(h)}_{\downarrow h \to 0}$$

(*) $$\boxed{\frac{dP(t)}{dt} = -\alpha P(t)}$$

Solving this equation for the initial condition $P(0)=1$ (why?) gives

$$\boxed{P(t) = e^{-\alpha t}}$$

Note:

The i-model is fully specified by the i-states and the i-processes.

The probability $P(t) = e^{-\alpha t}$ of an individual still being alive at time $t > 0$ is a derived property of the model.

Likewise, we can derive the probability distribution of the lifetime T of an individual.

$$\mathrm{Prob}\{T \leq t\} = 1 - P(t) = 1 - e^{-\alpha t}$$

is the cumulative distribution function of the random variable T.
Taking the derivative with respect to t gives the probability density:

$$\varphi(t) = \frac{d}{dt}\mathrm{Prob}\{T \leq t\} = \alpha e^{-\alpha t}, \quad t \geq 0.$$

which is the prob. dens. of the exponential distr. with mean $\frac{1}{\alpha}$.

So, given the i-model, the life time of an individual is exponentially distr. with an expected lifetime of length $\frac{1}{\alpha}$.

How to lift this to the population level?

The Law of the Large Numbers (LLN)
(strong version).

$R_1, R_2, \dots$ i.i.d. random variables with mean μ and finite variance.

Then:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} R_i = \mu \quad \text{a.s.}$$

(i.e., the mean of a sample of size n converges to the distribution mean with probability one as $n \to \infty$)

Now, let

$$R_i(t) = \begin{cases} 1 & \text{if individual } i \text{ is alive at time } t>0. \\ 0 & \text{otherwise.} \end{cases}$$

Then

(*) $\mu(t) := \mathcal{E}\{R_i(t)\} = P(t) = e^{-at}$

is the prob. that individual i is still alive at time t, and $\boxed{\frac{1}{n}\sum_{i=1}^{n} R_i(t)}$ is the proportion of individuals that are still alive at time t in a pop. of size n.

The LLN implies that

$$\boxed{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} R_i(t) = e^{-\alpha t} \quad \text{a.s}}$$

where we used (*) on page 8.

Let $\boxed{x(t)}$ be the pop. density of living individuals at time t in a volume of space (or area) of size V.

We have:

$\boxed{n = x(0)\,V}$ and $\boxed{\sum_{i=1}^{n} R_i(t) = x(t)\,V}$

so that $n \to \infty$ as $V \to \infty$ and we keep pop density constant.

And so

$$\frac{x(t)}{x(0)} = \frac{1}{n}\sum_{i=1}^{n} R_i(t) \xrightarrow{\text{a.s.}} e^{-\alpha t}$$

as we "scale up the system" by letting $V \to \infty$.

And that is how we lift the i-model to the population level, i.e., by "scaling up the system" so that the proportion of individuals alive at time t becomes equal to the probability of a given individual being alive at time t.

As we "scale up the system" the number of individuals alive at any point in time becomes infinite. That is why we "change variables" from <u>number of individuals</u> to <u>population</u> density $x(t)$, which satisfies $\boxed{x(t) = x(0)e^{-\alpha t}}$

which is the solution of the differential equation

$$\boxed{\frac{dx}{dt} = -\alpha x}$$

with initial value $x(0)$.

<u>end of example</u>