MATHEMATICAL MODELING 2012
EXERCISES 7-9

7. Consider the linear system

$$\frac{dz}{dt} = Az \ , \quad A \in \mathbb{R}^{2\times 2}$$

(a) Show that if λ_1 and λ_2 are eigenvalues of A with corresponding eigenvectors b_1 and b_2, then

$$z(t) = \beta_1 b_1 e^{\lambda_1 t} + \beta_2 b_2 e^{\lambda_2 t}$$

for given constants $\beta_1, \beta_2 \in \mathbb{C}$ is a solution.

(b) Show that the eigenspaces $[b_1]$ and $[b_2]$ are invariant, i.e., if $z(0) \in [b_i]$ $(i = 1, 2)$ then $z(t) \in [b_i]$ $\forall t$. Sketch the phase plane portraits (i.e., the orbits) when the eigenvalues are real and $\lambda_1 < \lambda_2 < 0$ and $\lambda_1 < 0 < \lambda_2$ and $0 < \lambda_1 < \lambda_2$.

(c) Show that if the eigenvalues are complex, then the λ_1 and λ_2 are complex conjugates of one another, and so are the eigenvectors b_1 and b_2.

(d) Show that if the eigenvalues are complex, then for every real-valued solution, the scalars β_1 and β_2 in part (a) are complex conjugates, and hence

$$z(t) = e^{\mathrm{Re}(\lambda)t}\Big(\alpha_1 \cos\big(\mathrm{Im}(\lambda)t\big) + \alpha_2 \sin\big(\mathrm{Im}(\lambda)t\big)\Big)$$

for given constants $\alpha_1, \alpha_2 \in \mathbb{R}$, and where $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ are, respectively, the real and imaginary parts of the eigenvalues. Sketch the phase plane portraits for $\mathrm{Re}(\lambda) < 0 \,\&\, \mathrm{Im}(\lambda) \neq 0$ and for $\mathrm{Re}(\lambda) > 0 \,\&\, \mathrm{Im}(\lambda) \neq 0$.

8. Consider the epidemic model

$$\begin{cases} \frac{ds}{dt} = -\beta si \qquad\qquad +\delta r & \text{(susceptible but healthy)} \\ \frac{di}{dt} = +\beta si \ -\gamma i & \text{(infected)} \\ \frac{dr}{dt} = \qquad\qquad +\gamma i \ -\delta r & \text{(recovered and temporally immune)} \end{cases}$$

Interpret the different terms on the right hand side in terms of individual level processes. Reduce the dimensionality of the system using a conservation relation. Do a phase-plane analysis and determine the local stability of all equilibria. You may have to consider different cases depending on the parameter values.

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(9) Consider the epidemic model in exercise (8).

Embed the model in a larger context including birth and death, assuming: (i) equal birth rates for susceptible and recovered individuals alike; (ii) infected individuals do not give birth at all; (iii) newborns are always susceptible; (iv) death rates for all types of individuals are the same.

Show that if β, γ and δ are large compared to the birth and death rates (i.e., the decease dynamics are fast), then the total population density $n := s + i + r$ is a slow variable. Do a two-timescale analysis; in particular, give the differential equation for the slow variable n exclusively in terms of n itself.

Give a qualitative analysis of the slow dynamics. You may have to consider different cases depending on the parameter values.