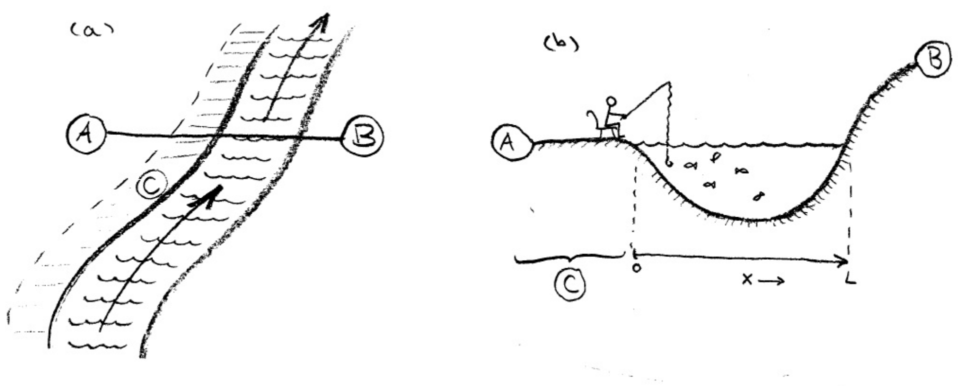
MATHEMATICAL MODELING 2012
EXERCISES 19-20

19.
Consider a situation as in the following figure: the picture on the left shows a map



of a river. The region on the left bank of the river has been labeled C. The picture on the right gives the cross section A–B, and it shows that the river is populated by fish, which are preyed upon by a terrestrial species living in compartment C.

In this exercise we formulate a spatial extension of the predator-prey model of Gause:

$$\begin{cases} \frac{dn}{dt} = g(n) - f(n)p & \text{(prey)} \\ \\ \frac{dp}{dt} = \gamma f(n)p & \text{(predator)} \end{cases}$$

(see page 36 of the lecture notes of 14–2–2012). To this end, let x denote the distance from the left shore as in the picture, and assume that individual fish move randomly. Assume further that compartment C is well-mixed.

(a) Give PDE/ODE equations describing the above situation.

(b) What boundary conditions would you impose on the fish density?

(c) What does it mean in terms of (the speed of) individual movement if we assume spatial structure for the fish but not for the predator?

(d) Re-formulate the model assuming that both the prey and the predator are spatially structured and allowing for both random movement and taxis in both populations.

20.
A grossly simplified model for the spread of panic in a crowd might look as follows:

let Q denote a quiet individual and P a panicking individual, and assume that the panic spreads through the reaction

$$\mathrm{P} + \mathrm{Q} \overset{\alpha}{\Longrightarrow} 2\mathrm{P}$$

and dissipates through the reaction

$$\mathrm{P} \overset{\beta}{\Longrightarrow} \mathrm{Q}$$

Also assume that Q and P move randomly with diffusion constants D_{Q} and D_{P}, respectively.

(a) Give the the corresponding PDE-s for the population densities of the P and the Q.

(b) Assuming reflecting boundaries on the bounded domain $[0, 1]$ for both the P and the Q, calculate under what conditions the panic can spread and keeps a hold on the population (i.e., analyze the stability of the panic-free situation). What conditions facilitate the spread of the panic.

(c) Same question as in **(b)**, but now with reflecting boundaries for the Q but absorbing boundaries for the P.

(d) How would your answers to questions **(b)** and **(c)** change if, in addition to the random movement of the P, we also assume that the P tend to move away from panicking crowds (i.e., if the P exhibit negative auto-taxis).

(e) Suppose that the reactions are fast compared to the spatial movement. Using time-scale separation, calculate the density-dependent diffusion coefficient for the total population density $N = P + Q$.