Appendix B

Theory of Poincaré and Bendixon

Consider the planar (i.e., 2-dim) system:

$\frac{dx}{dt} = f(x)$, $f: \mathbb{R}^2 \to \mathbb{R}^2$ cont. diff.

Some definitions:

- The forward orbit of x_0 is the set $\{x(t) : t \geq 0 \text{ and } x(0) = x_0\}$

  The ω-limit (omega-limit) is the set of limit points of the forward orbit.

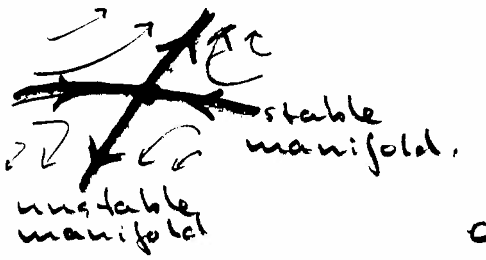
- The backward orbit of x_0 is the set $\{x(t) : t \leq 0 \text{ and } x(0) = x_0\}$

  The α-limit (alpha-limit) is the set of limitpoints of the backward orbit.

Examples

- If x_0 is an equilibrium, then the forward orbit of x_0, the backward orbit of x_0 and the corresponding ω-limit and α-limit are all the same, namely $\{x_0\}$.

- a stable node or focus is the ω-limit of all orbits converging to that node or focus.
- An unstable node or focus is the α-limit of all orbits starting in the vicinity of that node or focus.
- a saddle (in $\mathbb{R}^2$) has two stable manifolds and two unstable manifolds

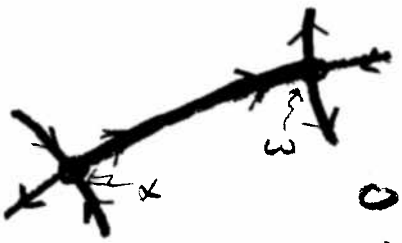


The saddle is the ω-limit of the stable manifolds and the α-limit of the unstable manifolds.

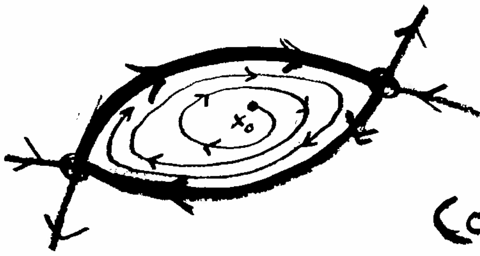
- a homoclinic orbit connects one of the unstable manifolds to one of the stable manifolds of the same saddle

The saddle then is both the ω-limit and the α-limit of the homoclinic orbit.

- A <u>heteroclinic orbit</u> connects an unstable manifold of one saddle to a stable manifold of another saddle. The ω-limit and the α-limit of a heteroclinic orbit are the respective saddles.



- A <u>heteroclinic cycle</u> consists of two (or more) heteroclinic orbits connecting two (or more) saddle as, for example, in the accompanying figure. Here the cycle is attracting from the inside, and so the ω-limit of the orbit of x_0 in the heteroclinic cycle, i.e, including the saddles.



- A <u>limit cycle</u> is an isolated periodic orbit. Like an equilibrium, a limit cycle is its own ω-limit and α-limit.

Theorem of Poincaré-Bendixon

The ω-limit of a bounded orbit in $\mathbb{R}^2$ is either a periodic orbit or contains at least one equilibrium.

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(We will not prove this theorem)

Corollary

If $D \subset \mathbb{R}^2$ is a closed and bounded trapping region(*) that does not contain an equilibrium, then D contains a periodic orbit.

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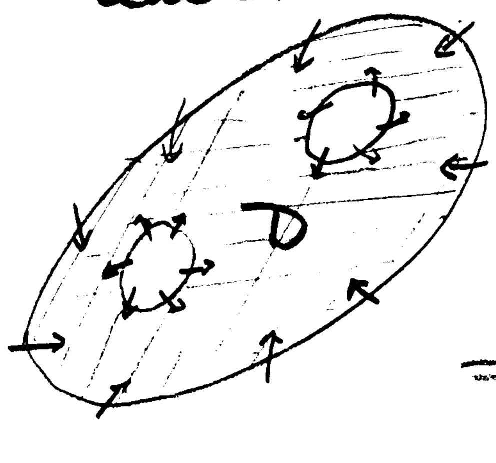
(footnote (*): D is forward invariant, i.e., orbits can enter but cannot leave.)



D is forward invar and does not contain an equilibrium.

$\Rightarrow$ D contains at least one periodic orbit.

Line integral of a vectorfield

$F : \mathbb{R}^2 \to \mathbb{R}^2$ cont. diff. vectorfield

$c : [a,b] \to \mathbb{R}^2$ cont. diff curve.

The integral of F over c is

$$\int_c F := \int_a^b F(c(t)) \cdot c'(t)\, dt \,.$$

↑ scalar product.

If we write $F = (F_1, F_2)$ and $c = (c_1, c_2)$, then we have

$$\int_c F = \int_a^b (F_1 c_1' + F_2 c_2')\, dt$$

Green's Theorem.

Let C be a positively oriented simple and cont. diff. closed loop, and let D be the region bounded by C. Then:

$$\int_C F = \int_D \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dx$$

Definition of divergence

The divergence of a vector field $f: \mathbb{R}^2 \to \mathbb{R}^2$ is

$$\operatorname{div} f := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

(i.e. $\operatorname{div} f = \operatorname{trace} \frac{df}{dx}$).

Bendixon's theorem.

Suppose $B \subset \mathbb{R}^2$ is simply connected (i.e, no holes), and $f: B \to \mathbb{R}^2$ is cont. diff. and such that either $\operatorname{div} f > 0$ almost everywhere on B or $\operatorname{div} f < 0$ almost everywhere on B.

Then the planar system

$$\frac{dx}{dt} = f(x)$$

does not have a periodic orbit in B.

Proof

Suppose that $\operatorname{div} f > 0$, almost everywhere.

To reach a contradiction, we assume that there exists a periodic orbit C with period $T > 0$, and let D be the region enclosed by C.

$$0 < \int_D \operatorname{div} f \, dx \overset{\text{def}}{=} \int_D \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx =$$

$$F := \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$$

$$= \int_D \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dx \overset{\text{Green}}{=} \int_C F =$$

$$\overset{\text{def}}{=} \int_C F(x(t)) \cdot \frac{dx}{dt} \, dt =$$

$$= \int_C \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dt =$$

$$= \int_C \underbrace{(-f_2 f_1 + f_1 f_2)}_{\equiv 0} \, dt = 0$$

which is a contradiction.

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Dulac's lemma.

Let $B \subset \mathbb{R}^2$ be simply connected and $f: B \to \mathbb{R}^2$ cont. diff, and suppose that there exists a strictly positive scalar-valued function $u: B \to (0, \infty)$ such that either $\operatorname{div}(uf) > 0$ almost everywhere on B, or $\operatorname{div}(uf) < 0$ almost everywhere on B

Then $\frac{dx}{dt} = f(x)$ does not have a periodic orbit in B.

Proof.

The "Dulac function" u does not affect the orbit, only the speed:

Define $\tau := \int_0^t \frac{ds}{u(x(s))}$ (non-linear scaling of time)

Then $\frac{dx}{d\tau} = u(x) f(x)$.

Next apply Bendixon's theorem.

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