Inverse problems for waves

Wave equation in $\mathbb{R}^n \times \mathbb{R}$ is

$$(\partial_t^2 - \Delta)u = \left(\frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\right)^2 \right) u(x,t) = 0$$

Motivation: $n=1$, functions

$$u(x,t) = h(x+t) + g(x-t)$$

Satisfy $(\partial_t^2 - \partial_x^2)u(x,t) = 0$

In $n \geq 2$, $\omega \in \mathbb{R}^n$, $|\omega| = 1$

$$u(x,t) = h(t - \omega \cdot x)$$

satisfy

$$(\partial_t^2 - \Delta)u(x,t) = h''(t - \omega\cdot x) - \sum_{j=1}^{n} (-\omega_j)^2 h''(t-\omega\cdot x) = 0$$

generally,

$$u(x,t) = \int_{S^{n-1}} F(t - x\cdot\omega; \omega)\, dS(\omega)$$

Satisfy $(\partial_t^2 - \Delta)u(x,t) = 0$

Let $F(x,t)$ denote sources of the wave. If $c(x)$ is the wave speed at $x \in \mathbb{R}^n$, wave equation takes the form

$$(1) \quad \begin{cases} \left(\partial_t^2 - c(x)^2 \Delta_x\right) u(x,t) = \dot{F}(x,t) \\ u(x,t)\big|_{t<0} = 0 \end{cases}$$

Let $\hat{u}(x,\omega)$ be the Fourier transform of $u(x,t)$ with respect to the time,

$$\hat{u}(x,\omega) = \int_{\mathbb{R}} e^{-i\omega t} u(x,t)\, dt.$$

then (1) yields

$$\left(-\omega^2 - c(x)^2 \Delta_x\right) \hat{u}(x,\omega) = \hat{F}(x,\omega)$$

or

$$\left(\Delta + n(x)\omega^2\right) \hat{u}(x,\omega) = \hat{G}(x,\omega),$$

$$n(x) = \frac{1}{c(x)^2}, \quad \hat{G} = -\frac{1}{c(x)^2}\hat{F}.$$

Let us consider the case when $c(x) = 1$

$$F(x,t) = \delta(x)\,\delta(t) = \delta_{(0,0)}(x,t)$$

$$\hat{F}(x,\omega) = \delta(x)\cdot 1$$

<u>Lemma</u> In $\mathbb{R}^3$,

$$g_\omega(x-z) = g_\omega(x,z) = \frac{1}{4\pi}\,\frac{e^{i\omega|x-z|}}{|x-z|}$$

satisfies

$$(\Delta + \omega^2)\, g_\omega(x,z) = \delta_z(x)$$

<u>proof</u> Let us consider the case $z=0$ and $\omega = 0$. Then

$$g_0(x) = \frac{1}{4\pi}\,\frac{1}{|x|}.$$

Now

$$\Delta\left(\frac{1}{|x|}\right) = \nabla\cdot\nabla\frac{1}{|x|} = -\nabla\cdot\left(\frac{x}{|x|^3}\right)$$

$$= -\frac{3}{|x|^3} + 3x\cdot\frac{x}{|x|^5} = 0 \quad , \quad x \neq 0$$

$$\frac{\partial}{\partial x_j}\frac{1}{|x|^a} = \frac{\partial}{\partial x_j}\frac{1}{(x\cdot x)^{a/2}} = -\frac{a}{2}(x\cdot x)^{-\frac{a}{2}-1}\,2x_j$$

$$= -a\,\frac{1}{|x|^{a+2}}\,x_j.$$

thus, if we consider $\frac{1}{|x|} \in L^1_{loc}(\mathbb{R}^3)$ as a distribution, for $\phi \in C_c^\infty(\mathbb{R}^3)$

$$\langle \Delta g_0(x), \phi \rangle = \langle \nabla \cdot \nabla g_0, \phi \rangle$$

$$= \langle g_0, \nabla \cdot \nabla \phi \rangle$$

$$= \int_{\mathbb{R}^3} \frac{1}{|x|} \nabla \cdot \nabla \phi(x)\, dx$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3 \setminus B(\varepsilon)} \frac{1}{|x|} \Delta \phi(x)\, dx$$

$$= \lim_{\varepsilon \to 0} \Bigg[\int_{\mathbb{R}^3 \setminus B(\varepsilon)} \overbrace{\Delta \frac{1}{|x|}}^{0} \cdot \phi(x)\, dx + \int_{\partial B(\varepsilon)} \frac{1}{|x|}\, n \cdot \nabla \phi(x)\, dS(x) - \int_{\partial B(\varepsilon)} \underbrace{n \cdot \nabla \frac{1}{|x|}}_{= -\frac{1}{|x|^2}} \cdot \phi(x)\, dS(x) \Bigg]$$

$$= \phi(0) = \langle \delta_0, \phi \rangle$$

$\square$

<u>Theorem</u> Let $F \in C_0^\infty(\mathbb{R}^3)$ be supported in $B(M) = \{x \in \mathbb{R}^3 ; |x| \leq M\}$

Then

$$v(x) = \int_{\mathbb{R}^3} g_w(x-z) F(z)\, dz$$

is solution of

(1) $$(\Delta + w^2) v(x) = F(x) \quad \text{in } \mathbb{R}^3$$

(2) $$\lim_{|x| \to \infty} \frac{1}{|x|} \left(\hat{x} \cdot \nabla - iw \right) v(x) = 0 .$$

Condition (2) is the Sommerfeld radiation condition.

<u>proof</u> $$v(x) = \int_{\mathbb{R}^3} g_w(z) F(x-z)\, dz .$$

thus

$$(\Delta_x + w^2) v(x) = \int_{\mathbb{R}^3} g_w(z) (\Delta_x + w^2) F(x-z)\, dz$$

$$= \int_{\mathbb{R}^3} g_w(z) (\Delta_z + w^2) F(x-z)\, dz$$

$$= \langle g_w , (\Delta_z + w^2) F(x - \cdot) \rangle$$

$$= \langle (\Delta_z + w^2) g_w \, ; \, F(x - \cdot) \rangle$$

$$= \langle \delta_0 \, , \, F(x - \cdot) \rangle = F(x)$$

this prove (1)

We leave (2) to an exercise based on fact that

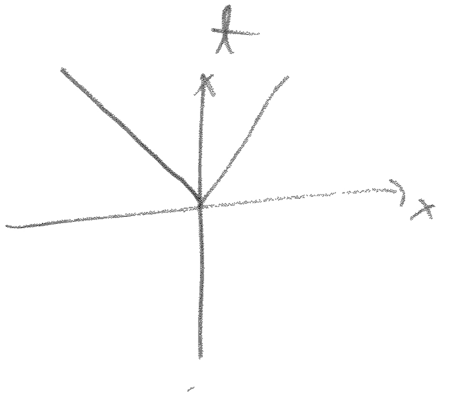
$$\lim_{r \to \infty} r^{-1} \left(\frac{\partial}{\partial r} - iw \right) \left(\frac{e^{iwr}}{r} \right) = 0$$

$\square$

(2) is so-called radiation condition.

Note that

$$\mathcal{F}_{t \to w} \left(\frac{\delta(t - |x|)}{4\pi |x|} \right) = \frac{e^{i|x|w}}{4\pi |x|} .$$



Lemma Let $F \in C_0^\infty(\mathbb{R}^3)$, $\mathrm{Supp}(F) \subset B(M)$. Then

$$u(x) = \int_{\mathbb{R}^3} g_\omega(x-z)\, F(z)\, dz$$

satisfies

$$\lim_{r\to\infty} \left(\frac{e^{i\omega r}}{4\pi r}\right)^{-1} u(r\hat{x})$$

$$= u_\infty(\hat{x}) := \int_{\mathbb{R}} e^{-i\omega \hat{x}\cdot z} F(z)\, dz,$$

when $\hat{x} = \frac{x}{|x|}$

When $x \to \infty$, we see that for $z \in \mathbb{R}^3$

$$|x-z|^2 = |x|^2 - 2x\cdot z + |z|^2.$$

$$= |x|^2 \left[1 - 2\frac{x}{|x|^2}\cdot z + \frac{|z|^2}{|x|^2}\right]$$

As

$$(1+t)^{1/2} = 1 + \frac{1}{2}t + \mathcal{O}(t^2), \quad t\to 0$$
$$(1+t)^{-1} = 1 - t + \mathcal{O}(t^2)$$

when $\mathcal{O}(f(t)) \le C|f(t)|$

we have for any $z \in \mathbb{R}^3$, $|z| \le M$)

$$|x-z| = |x|\cdot\left[1 - \frac{x}{|x|^2}\cdot z + \mathcal{O}_M\left(\frac{1}{|x|^3}\right)\right]$$

$$= |x| - \hat{x}\cdot z + \mathcal{O}_M\left(\frac{1}{|x|}\right), \quad |x|\to\infty$$

when $\hat{x} = \frac{x}{|x|} \in S^1$ and $\left|\mathcal{O}\left(\frac{1}{|x|}\right)\right| \le \frac{C(M)}{|x|}$.

thus

$$\frac{1}{|x-z|} = \frac{1}{|x|}\cdot\frac{1}{1 - \frac{1}{|x|}\hat{x}\cdot z + \mathcal{O}_M\left(\frac{1}{|x|}\right)}$$

$$= \frac{1}{|x|}\left(1 + \mathcal{O}_M\left(\frac{1}{|x|}\right)\right)$$

and

$$\frac{e^{i\omega|x-z|}}{4\pi|x-z|} = \frac{e^{i\omega\left(|x| - \hat{x}\cdot z + \mathcal{O}_M\left(\frac{1}{|x|}\right)\right)}}{4\pi|x|\left(1+\mathcal{O}_M\left(\frac{1}{|x|}\right)\right)}$$

$$= \frac{e^{i\omega|x|}}{4\pi|x|}\left[e^{-i\omega\hat{x}\cdot z}\right]\cdot\left(1+\mathcal{O}_M\left(\frac{1}{|x|}\right)\right).$$

Note that in the above computation, we have that for all M there is C_M s.t.

$$\left|\mathcal{O}_M\left(\frac{1}{|x|}\right)\right| \le \frac{C_M}{|x|}$$

for all $|z| \le M$.

Thus

$$u(x) = \int_{\mathbb{R}^3} \frac{e^{i\omega|x|}}{4\pi|x|} e^{-i\omega\hat{x}\cdot z} F(z)\,dz + \mathcal{O}_M\left(\frac{1}{|x|^2}\right)$$

$$= \frac{e^{i\omega|x|}}{4\pi|x|} u_\infty(\hat{x}) + \mathcal{O}_M\left(\frac{1}{|x|^2}\right)$$

$\square$

Here $u_\infty(\hat{x})$ is called the Far field of $u(x)$.

Inverse source problem:

Given $u_\infty(\hat{x})$, can we find $F(x)$?

Answer no: we can find only

$$u_\infty(\hat{x}) = \hat{F}(w\hat{x}), \quad \hat{x} \in S^2.$$

Example of invisible sources:

Let

$$u_0(x) \in C_0^\infty(\mathbb{R}^3), \quad \mathrm{supp}(u_0) \subset B(M)$$

$$F_0(x) = (\Delta + k^2) u_0(x) \in C_0^\infty(\mathbb{R}^3)$$

then

$$(\Delta + k^2) u_0 = F_0$$

$$u_0 \big|_{\mathbb{R}^3 \setminus B(M)} = 0$$

$$(u_0)_\infty(\hat{x}) = 0 \quad \text{for all } \hat{x} \in S^2.$$

Inverse problem for $n(x)$.

Assume that $\frac{1}{c(x)^2} = n(x) = 1 + m(x)$,

$m(x) > -1$, $m \in C_0^\infty(\mathbb{R}^3)$.

Let us consider solutions of

$$(3)\quad \begin{cases} (\Delta + \omega^2 n(x))\, u(x, \omega, \theta) = 0 \\ u(x, \omega, \theta) = u_{in}(x, \omega, \theta) + u_{sc}(x, \omega, \theta) \\ u_{sc} \text{ satisfies Sommerfeld radiation condition} \\ u_{in}(x, \omega, \theta) = e^{i\omega\theta\cdot x} \end{cases}$$

Motivation:

$$\mathcal{F}_{t\to\omega}\left(\delta(t - \theta\cdot x)\right) = e^{i\omega\theta\cdot x}$$

Let

$$u^{sc}_{\infty}(\hat{x},\theta,\omega) = \lim_{r\to\infty} \left(\frac{e^{i\omega r}}{4\pi r}\right)^{-1} u^{sc}(r\hat{x},\theta,\omega).$$

<u>Inverse problem</u>: Assume we know $u^{sc}_{\infty}(\hat{x},\theta,\omega)$ for $\hat{x},\theta\in S^2$, $\omega \geq 0$. Can we find $m(x)$?

Answer is yes, but this problem is quite difficult. Because of this, we consider the linearized problem:

1) Let $A_\omega v = g_\omega * v$. Then (3) yields

$$\begin{cases} (\Delta+\omega^2)u^{sc} = -m\omega^2(u^{in}+u^{sc}) \\ u^{sc} \text{ satisfy S.R.C.} \end{cases}$$

and thus

$$u^{sc} = -A_\omega\left[m\omega^2(u^{in}+u^{sc})\right]$$

If $m(x) = \varepsilon m_0(x)$, we have

$$u^{sc} = -\varepsilon A_\omega\left[m_0\omega^2(u^{in}+u^{sc})\right]$$

$$= -\varepsilon A_\omega(m_0\omega^2 u^{in}) + \varepsilon^2 A_\omega\left[m_0\omega^2 u^{in}\right] + \varepsilon^3 \ldots$$

Assuming that ε is small, and approximating that $\mathcal{O}(\varepsilon^2) \approx 0$ we define linearized solutions

$$u^{lin}(x) := -A_\omega(m\omega^2 u^{in})$$

<u>Note</u>: If $K : m \mapsto u^{sc}$ is the direct map, then its Frechet derivative at zero is

$$DK\big|_0 : m \to u^{lin}$$

Now

$$(4)\begin{cases}(\Delta+\omega^2)\,u^{lin}(x) = -m(x)\,\omega^2\,u^{in}(x) & \text{in } \mathbb{R}^3\\ u^{lin} \text{ satisfies S.R.C} & \end{cases}$$

Linearized inverse problem:

Theorem Assume that we are given $u^{lin}_\infty(\hat{x},\theta,\omega)$ for all $\hat{x},\theta\in S^2$, $\omega\geq 0$. Then we can uniquely determine $m(x)$.

proof. By (4), denoting

$$F(x) = -m(x)\,\omega^2 u^{in}(x,\theta,\omega),$$
$$u^{in}(x,\theta,\omega) = e^{i\omega\theta\cdot x}$$

we have

$$u_\infty^{lin}(\hat{x};\theta,\omega) = \int_{\mathbb{R}^3} e^{-i\omega\hat{x}\cdot z}\, F(z)\,dz$$

$$= -\omega^2 \int_{\mathbb{R}^3} e^{-i\omega\hat{x}\cdot z}\, e^{i\omega\theta\cdot z}\, m(z)\,dz$$

$$= -\omega^2 \int_{\mathbb{R}^3} e^{-i[\omega(\hat{x}-\theta)]\cdot z}\, m(z)\,dz$$

Thus

$$-\omega^{-2} u_\infty^{lin}(\hat{x},\theta,\omega) = \hat{m}(\omega(\hat{x}-\theta))$$

Let $\xi \in \mathbb{R}^3 \setminus 0$ and choose $\hat{x} = -\theta = \frac{\xi}{|\xi|}$, $\omega = |\xi|/2$. Then $\hat{m}(\xi) = \hat{m}(\omega(\hat{x}-\theta))$.
Clearly, $\hat{m}(0) = \lim_{\omega\to 0} \hat{m}(\omega(\hat{x}-\theta))$.

Thus we can find $\hat{n}(\rho)$ for all $\rho \in \mathbb{R}^3$ and

$$n(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\rho}\, \hat{n}(\rho)\, d\rho$$

$\square$

Note: Above proofs shows that the linearized backscattering data

$$u_\infty^{lin}(\hat{x}, -\hat{x}, \omega), \quad \hat{x} \in S^2, \omega \geq 0$$

determine $n(z)$ uniquely.