Exact ground states of spin chains from quantum Knizhnik–Zamolodchikov equation

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starting grant "LIC"

work in collobaration with P. Di Francesco, A. Razumov, Yu. Stroganov, T. Fonseca.

Objective

- Provide explicit expressions in finite size for the ground state of spin chains/transfer matrices of 2D stat mech models (as a first step towards computation of physically interesting quantities).
- Even if the models are integrable, this is in general impossible $\longrightarrow$ consider special points of the parameter space.
- Here we shall try to *explain* the work of PZJ and collaborators on vertex/loop models, in order to extend it. The key equation is the quantum Knizhnik–Zamolodchikov equation.
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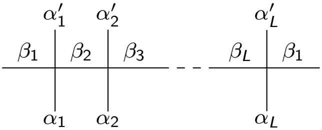
Transfer matrix

Row-to-row transfer matrix T of a statistical model on the two-dimensional square lattice with PBC:

$$\langle \alpha' | \, T \, | \alpha \rangle = \sum_{\beta_1,\ldots,\beta_L}$$



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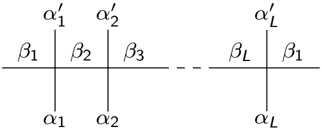

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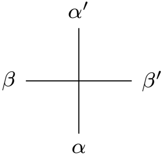

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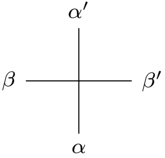

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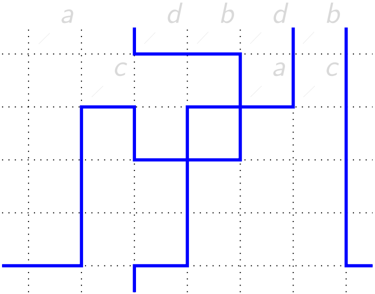
Example: Six- and Eight-vertex models

A configuration of the eight-vertex model:



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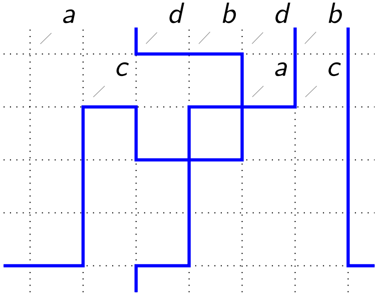

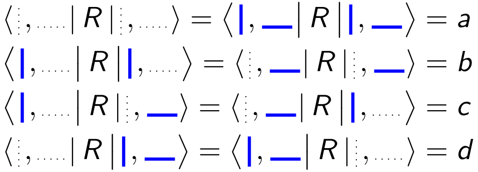

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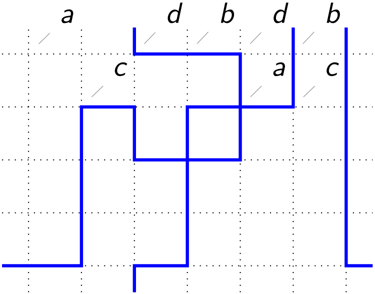

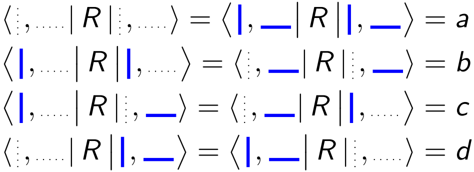

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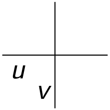
$R(u/v) =$ 

such that certain equations are satisfied (YBE).

The fully inhomogeneous twisted transfer matrix is

$$T(u|z_1,\ldots,z_L) = \mathrm{tr}(R_{0L}(u/z_L)\ldots R_{01}(u/z_1)\Omega)$$

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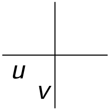
$R(u/v) =$ 

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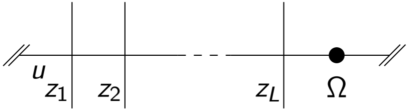
$=$ 

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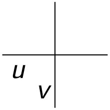
$R(u/v) =$ 

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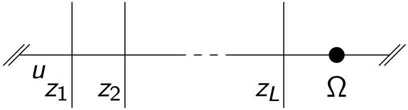
$=$ 

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Quantum integrability

For fixed inhomogenities and varying spectral parameter, the transfer matrices commute:

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Ex: spin 1/2

$$H = -\frac{1}{2}\sum_{i=1}^{L}\left(J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z\right)$$

where the $\sigma_i^{x,y,z}$ are Pauli matrices acting on the i^{th} factor of $\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$. $(L+1 \equiv 1)$

- the XXX chain (rational 6-vertex): $J_x = J_y = J_z$ [Heisenberg, 1928; Bethe, 1931]
- the XXZ chain (6-vertex): $J_x = J_y \neq J_z$ [Yang and Yang, 1966]
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The diagonalization problem

The first problem in the study of these models is the diagonalization of the transfer matrix/Hamiltonian.

We focus on the trigonometric case (e.g., XXZ). A standard method is the (algebraic) Bethe Ansatz:

$$|\Psi\rangle = B(u_1)\cdots B(u_n)\,|\varnothing\rangle$$

where the u_i satisfy Bethe equations.

In general these equations are not solvable for finite L. One can only obtain explicit formulae for the ground state (and excitations above it) in the thermodynamic limit where $L \to \infty$.

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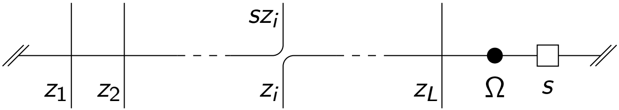
Quantum Knizhnik–Zamolodchikov equation

How about solving a different linear problem? The quantum Knizhnik–Zamolodchikov equation is a system of holonomic q-difference equations which appears

- in the study of form factors of integrable models [Smirnov, '86]
- in the representation theory of quantum affine algebras [Frenkel, Reshetikhin '92]
- in the study of correlation functions of integrable models [Jimbo, Miwa et al, '93]
- in relation to representation theory of affine Hecke algebra and DAHA [Cherednik, Pasquier, '90s]

Define the operator for $i = 1, \ldots, L$:

$A_i(z_1, \ldots, z_L) =$



where s is a new variable, usually parameterized as $s = q^{2(\ell+2)}$; ℓ is the level.

The quantum Knizhnik–Zamolodchikov equation is the set of equations:

$$A_i \Psi(z_1, \ldots, z_L) = \Psi(z_1, \ldots, s\, z_i, \ldots, z_L) \qquad i = 1, \ldots, L$$

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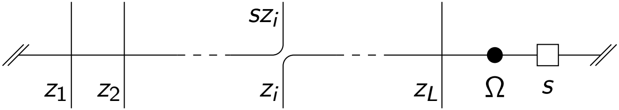

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Quasi-classical limit of qKZ

What happens when $s \to 1$? Set $s = e^{\hbar}$. Then [Reshetikhin, '93]

$$|\Psi\rangle = e^{S/\hbar} |\Psi_0\rangle \left(1 + O(\hbar)\right)$$

where S is a scalar function.

Expanding qKZ at first order in $\hbar$, one finds:

$$A_i |\Psi_0\rangle = \exp\left(z_i \frac{\partial}{\partial z_i} S\right) |\Psi_0\rangle$$

i.e., $|\Psi_0\rangle$ is eigenvector of the A_i (and therefore, of the transfer matrix $T(u)$) with eigenvalue $\exp(z_i \frac{\partial}{\partial z_i} S)$.

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Solution of qKZ

One way to solve the qKZ equation is to use off-shell Bethe Ansatz:

$$|\Psi\rangle = \sum_{u_1,\ldots,u_n} f(u_1,\ldots,u_n|z_1,\ldots,z_L)B(u_1)\ldots B(u_n)\,|\varnothing\rangle$$

(off-shell means the u_i do not satisfy Bethe equations).

This leads to rather cumbersome expressions. Furthermore, in the limit $s \to 1$, sums turn into integrals, which are dominated by a saddle point, which turns out to be nothing but Bethe equations.

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$$\Psi_{\alpha_1,\ldots,\alpha_L}(z_1,\ldots,z_L) = \langle 0|\,\Phi_{\alpha_1}(z_1)\ldots\Phi_{\alpha_L}(z_L)\,|0\rangle$$

(q-deformation of conformal blocks of WZW; path indices implied).

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Punch line

If the level is a positive integer ℓ, and $\Phi(z)$ is a so-called "perfect vertex operator" (i.e., for $U_q(\widehat{\mathfrak{sl}(2)})$, of spin $\ell/2$, $d = \ell + 1$), then

- There is only one conformal block (single path).
- Ψ satisfies a simpler form of the qKZ equation, namely Smirnov's qKZ system.
- Ψ has a simple analytic behavior – a product of two-body factors times a vector-valued polynomial in $z_1, \ldots, z_L, q, q^{-1}$.
- Ψ has a smooth limit as $s \to 1$.
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General idea: use (q-deformed) bosonization technique.

Here we have a (q-deformed) chiral WZW model at level ℓ.

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Construction of bosonic vertex operators

Consider bosonic modes:

$$[b_m, b_n] = \frac{[2m][m]}{m}\delta_{m,-n} \qquad m,n \in \mathbb{Z} \qquad [b_0,\beta] = 2$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$ and form the VO:

$$\Phi^{(0)}(z) = e^{\frac{\ell}{2}\beta} z^{\frac{1}{2}b_0} : e^{-\sum_{n\neq 0} q^{\ell|n|/2} \frac{b_n}{[2n]} z^{-n}} :$$

Roughly, $\Phi^{(0)}(z) \approx \, : e^{\sqrt{\ell/2}\,\phi(z)} :$ where $\phi(z)$ is a bosonic field.

Construction of q-parafermions

- These q-parafermions are nonlocal fields $\psi^i(z)$, $i \in \mathbb{Z}/\ell\mathbb{Z}$, $\psi^0 = 1$.
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Putting everything together

The whole multiplet of the VO can be written as

$$\Phi^{(b)}(z) \propto \oint \prod_{i=1}^{b} \frac{u_i du_i}{(u_i - q^\ell z)(z - q^\ell u_i)} : f(u_1) \dots f(u_b) \Phi^{(0)}(z) :$$

where the current is

$$f(z) = \psi(z)\, e^{-\beta} z^{-\frac{1}{\ell} b_0} : e^{\sum_{n \neq 0} q^{\ell |n|/2} \frac{b_n}{[\ell n]} z^{-n}} :$$

This requires knowing correlations of the q-parafermions:

$$M(u_1, \dots, u_{n\ell}) := \langle 0 | \, \psi(u_1) \dots \psi(u_{n\ell}) \, | 0 \rangle$$

$M(u_1, \dots, u_{n\ell})$ turns out to be the Macdonald polynomial associated to the partition $(\underbrace{2(n-1), \dots, 2(n-1)}_{\ell}, \dots, \underbrace{2, \dots, 2}_{\ell})$ and to parameters $\mathbf{t} = q^{-2}$ and $\mathbf{q} = q^{2(\ell+1)}$.

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The final formula

Call (a_i) the sequence of indices of lowering operators, i.e.,

$$\Psi_{a_1,\ldots,a_{n\ell}} := \langle 0|\, \Phi^{(b_1)}(z_1) \ldots \Phi^{(b_L)}(z_L)\, |0\rangle$$

where $b_k = \#\{i : a_i = k\}$. Assume the size is even, $L = 2n$. (there is an analogue in odd size)

Then

$$\Psi_{a_1,\ldots,a_{n\ell}} = \prod_{i<j}^{2n} \prod_{k=1}^{\ell} (q^{2k} z_i - z_j) \oint \ldots \oint \prod_{i=1}^{n\ell} u_i du_i$$

$$\times \frac{M(u_1, \ldots, u_{n\ell}) \prod_{i<j}(u_i - u_j)}{\prod_{i<j}(u_i - q^2 u_j) \prod_{j \leq a_i}(q^{-s} u_i - q^s z_j) \prod_{j \geq a_i}(q^s u_i - q^{-s} z_j)}$$

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Spin chains

- At $q = -e^{-i\pi/(\ell+2)}$, this formula provides the exact ground state of the integrable spin $\ell/2$ chain with a diagonal twist $\Omega = \mathrm{diag}(q^\ell, q^{\ell-2}, \ldots, q^{-\ell})$.
- At $\ell = 1$, we recover the results of [Razumov, Stroganov, PZJ '07] on the XXZ chain at $\Delta = -1/2$.
  - In this case, the integral formula provides deep connections with combinatorics.
  - For example, in odd size $L = 2n + 1$, $\Psi_{max} = A_n$ (number of Alternating Sign Matrices of size n).
  - This led to a proof of the doubly refined Alternating Sign Matrix conjecture [Fonseca, PZJ '08].
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Relation to loop models

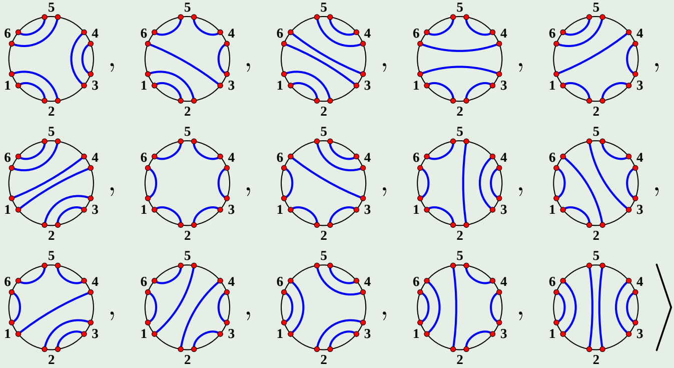
The twist $\Omega = \mathrm{diag}(q^\ell, q^{\ell-2}, \dots, q^{-\ell})$ is exactly the one for which there is an equivalent formulation in terms of a loop model:

$$\mathcal{H}_{\ell,n}^{\text{loop model}} \subset \mathcal{H}_{\ell,n}^{\text{spin chain}} = (\mathbb{C}^{\ell+1})^{\otimes 2n}$$

$\mathcal{H}_{\text{loop model}}$ has a basis of "fused link patterns", i.e., planar pairings of $2n\ell$ points such that successive groups of size ℓ contain no internal pairings.

Example

$$\mathcal{H}_{2,6} = \Big\langle$$



$$\Big\rangle$$

$$H = \sum_{i=1}^{2m} \sum_{j=1}^{\ell} \frac{q - q^{-1}}{q^j - q^{-j}} e_i^{(j)}$$

where

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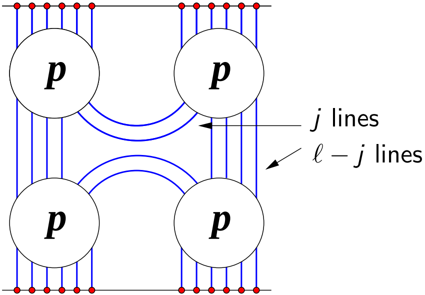

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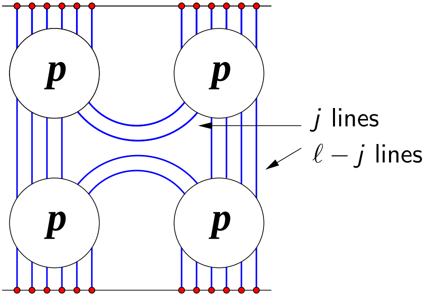
$e^{(j)} =$ 

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