An overview of recent results concerning low temperature Gibbs measures for logartihmically correlated Gaussian fields

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The logarithmically correlated REM

Consider a model for a particle in a Gaussian random potential with logarithmic correlations (disordered system).

The Gibbs measure is

$$\mu_{\beta,N}(r) = \frac{1}{Z_{\beta,N}} e^{-\beta V_N(r)},$$

where $r \in \{-N, ..., N\}^d \subset \mathbb{Z}^d$, β is the inverse temperature, $E((V_N(r) - V_N(r'))^2) \sim \log |r - r'|$ (long range behavior) and $E(V_N(r)^2) \sim \log N$.

Related to many different physical and mathematical problems.

The logarithmically correlated REM

Expected from non-rigorous arguments and numerics (Carpentier and Le Doussal): As $N \to \infty$

- There is a phase transition in the model.
- Universality: only the long range behavior of the correlations is important.
- For large β, μ_β is supported on only a few configurations - the low temperature measure is 'atomic'.
- For small β, μ_β is more evenly distributed.
- The low temperature regime is frozen - quantities like the free energy density and certain generating functions become independent of β - similarly to the classical REM with no correlations (introduced by Derrida).

Liouville quantum gravity

Let ϕ be the GFF on a domain $D \subset \mathbb{C}$: Gaussian random distribution with covariance $G_D(z,w) \sim \log \frac{1}{|z-w|}$ (short range behavior).

For $\phi_\epsilon =$ the GFF with a cutoff at scale ϵ, let

$$\mu_{\epsilon,\beta}(d^2z) = \epsilon^{\frac{\beta^2}{2}} e^{\beta\phi_\epsilon(z)} d^2z.$$

- $\mu_{\epsilon,\beta} =: e^{\beta\phi_\epsilon(z)} : d^2z$ (normal ordering).
- $\mu_{\epsilon,\beta}$ converges weakly almost surely as $\epsilon \to 0$ (continuum limit) for $\beta < 2$ (Duplantier and Sheffield).
- The limit is the 'quantum area measure'.

The measures μ_β are related to issues like the KPZ relation (Duplantier and Sheffield) and conformal welding (Astala, Jones, Kupiainen and Saksman; Sheffield).

- What about $\beta \geq 2$?
- For a hierarchical approximation of the field, it turns out that with normal ordering, the measure converges to a zero measure.
- With a more subtle (deterministic) normalization of the measure, there is convergence in distribution to something non-trivial for $\beta \geq 2$ as well.

A hierarchical model - multiplicative cacasdes

- On a binary tree, put i.i.d. standard Gaussians on each edge.
- For each branch of length n (indexed by $k = 1, ..., 2^n$) define X_k^n to be the sum of the Gaussians along that branch (a special case of a branching random walk).
- Define a metric on the set of branches of length n: $d_n(k, k') = N(k, k')$, where $N(k, k')$ is the number of levels the branches k and k' differ on.
- $E((X_k^n - X_{k'}^n)^2) \sim d_n(k, k')$.
- $E((X_k^n)^2) \sim \log 2^n$

A hierarchical model - multiplicative cascades

- Identify the branch k with the interval $[(k-1)2^{-n}, k2^{-n})$ and define a Gaussian field $\{X_z^n\}_{z\in[0,1]}$ through this identification.
- Sublogarithmic correlations: $E((X_z^n - X_{z'}^n)^2) \gtrsim \log \frac{1}{|z-z'|}$.
- Approximation to a logarithmically correlated field, though slightly weaker correlations and not translation invariant.

A hierarchical model - multiplicative cascades

- Define the 'discrete quantum measure' on $[0,1]$: $\mu_{n,\beta}(dz) = 2^n e^{-\beta X_z^n} dz$ (the 2^n is put here just to simplify some notation later on).
- Such measures were first introduced by Mandelbrot.
- Kahane proved that for $c(\beta) = \frac{\beta}{2} + \frac{\log 2}{\beta}$, $e^{-\beta c(\beta) n}\mu_{n,\beta}$ converges weakly almost surely as $n \to \infty$ to an atomless measure when $\beta < \sqrt{2\log 2}$ (normal ordering as for the GFF).
- A key part of the proof is that the total mass of $\mu_{n,\beta}$ ($||\mu_{n,\beta}|| = Z_{n,\beta} = \sum_k e^{-\beta X_k^n}$) is a positive martingale after a deterministic normalization (multiplication by $e^{-\beta c(\beta) n}$) and converges to a positive random variable for $\beta < \sqrt{2\log 2}$.
- For $\beta \geq \sqrt{2\log 2}$ this martingale converges to 0 almost surely.
- First task: try to find a suitable normalization in the $\beta \geq \sqrt{2\log 2}$ case.

A recursion relation for a generating function

- By the self similar tree structure of the field

$$Z_{n,\beta} \overset{d}{=} e^{-\beta V}(Z_{n-1,\beta}^{(1)} + Z_{n-1,\beta}^{(2)}),$$

  where V is a standard Gaussian, $Z_{n-1,\beta}^{(i)}$ are independent copies of $Z_{n-1,\beta}$ which are independent of V.
- Following Derrida and Spohn, define the generating function $G_{n,\beta}(x) = E(\exp(-e^{-\beta x} Z_{n,\beta}))$ (simply a clever way to parametrize the Laplace transform).
- Applying the decomposition of $Z_{n,\beta}$ to the generating function we get

$$\begin{aligned} G_{n+1,\beta}(x) &= E(\exp(-e^{-\beta(x+V)}(Z_{n,\beta}^{(1)} + Z_{n,\beta}^{(2)}))) \\ &= \int \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} E(\exp(-e^{-\beta(x+y)} Z_{n,\beta}))^2 dy \\ &= \int \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} G_{n,\beta}(x+y)^2 dy. \end{aligned}$$

The continuum time formulation and the KPP equation

As noted by Derrida and Spohn, if one considers a branching Brownian motion instead of a branching random walk, one ends up with the KPP equation:

$$\frac{\partial}{\partial t} G_{t,\beta}(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} G_{t,\beta}(x) + G_{t,\beta}(x)^2 - G_{t,\beta}(x),$$

with initial data $G_{0,\beta}(x) = \exp(-e^{-\beta x})$.

- Such equations are known to have moving front solutions: $G_t(x) = G(x - ct)$.
- Bramson has proven that in one defines $m_\beta(t) = G_{t,\beta}^{-1}(\frac{1}{2})$, $G_{t,\beta}(x + m_\beta(t))$ converges uniformly to a function w_β satisfying

$$\frac{1}{2} w_\beta'' + \tilde{c}(\beta) w_\beta' + w_\beta^2 - w_\beta = 0.$$

The continuum time formulation and the KPP equation

- Here $\tilde{c}(\beta) = \frac{\beta}{2} + \frac{1}{\beta}$ for $\beta \leq \sqrt{2}$ and $\tilde{c}(\beta) = \tilde{c}(\sqrt{2}) = \sqrt{2}$ for $\beta \geq \sqrt{2}$.
- $\tilde{c}(\beta)$ and w_β 'freeze' at low temperatures.
- For $\beta < \sqrt{2}$, $1 - w_\beta(x) \sim e^{-\beta x}$ as $x \to \infty$ and for $\beta = \sqrt{2}$, $1 - w_\beta(x) \sim xe^{-\sqrt{2}x}$ as $x \to \infty$.
- Bramson was able to extract the asymptotic behavior of $m_\beta(t)$ as well:
    - For $\beta < \sqrt{2}$, $m_\beta(t) - \tilde{c}(\beta)t$ converges as $t \to \infty$.
    - For $\beta = \sqrt{2}$, $m_\beta(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \log t + \mathcal{O}(1)$.
    - For $\beta > \sqrt{2}$, $m_\beta(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + \mathcal{O}(1)$.

Back to discrete time

Many tools used to analyze the KPP equation work for the discrete case as well and one can prove the corresponding results.

Theorem

For $m_{\beta,n} = G_{n,\beta}^{-1}(\frac{1}{2})$, $G_{n,\beta}(x + m_{\beta,n})$ converges uniformly to a function g_β satisfying

$$g_\beta(x) = \int \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} g_\beta(x + y + \hat{c}(\beta))^2 dy.$$

Here $\hat{c}(\beta) = c(\beta) = \frac{\beta}{2} + \frac{\log 2}{\beta}$ for $\beta \leq \beta_c = \sqrt{2 \log 2}$ and $\hat{c}(\beta) = \hat{c}(\beta_c)$ for $\beta \geq \beta_c$.

Back to discrete time

- $\hat{c}(\beta)$ and g_β freeze at low temperatures.
- $1 - g_\beta(x) \sim e^{-\beta x}$ as $x \to \infty$ for $\beta < \beta_c$ and $1 - g_\beta(x) \sim xe^{-\beta_c x}$ as $x \to \infty$ for $\beta = \beta_c$.
- The integral equation has a unique solution (up to translations) if we assume some regularity on g_β (e.g. g_β increasing from 0 to 1).

Convergence of the total mass

$m_{\beta,n}$ has a similar form as in the continuum time case:

Theorem

- *For* $\beta < \beta_c$, $m_{\beta,n} - \hat{c}(\beta)n$ *converges as* $n \to \infty$.
- *For* $\beta = \beta_c$, $m_{\beta,n} = \sqrt{2\log 2}n - \frac{1}{2\sqrt{2\log 2}} \log n + \mathcal{O}(1)$.
- *For* $\beta > \beta_c$, $m_{\beta,n} = \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}} \log n + \mathcal{O}(1)$.

Since $G_{n,\beta}$ was essentially the Laplace transform, these convergence results imply that $e^{-\beta m_{\beta,n}} Z_{n,\beta}$ converges in distribution (and the Laplace transform of the limiting distribution is g_β up to a change of variable).

Convergence of the total mass

Similar convergence results have been obtained by Madaule and Aidekon and Shi. In fact they prove convergence for more general branching random walks and they show that the $\mathcal{O}(1)$ term can be neglected. Moreover, Aidekon and Shi prove that the convergence at $\beta = \beta_c$ is in probability.

β dependence of M_β and stable subordinators

If we write M_β for the limit, the distributional recursion of $Z_{n,\beta}$ implies that

$$M_\beta \stackrel{d}{=} e^{-\beta(V+\hat{c}(\beta))}(M_\beta^{(1)} + M_\beta^{(2)}).$$

- Such equations have been studied (Durrett and Liggett - properties of $E(e^{-uM_\beta})$; Buraczewski - properties of $P(M_{\beta_c} \geq x)$).
- Let $f(u,\beta) = E(e^{-uM_\beta})$ be the Laplace transform of the law of M_β (so $f(e^{-\beta x},\beta) = g_\beta(x)$).
- For $\beta \geq \beta_c$, $g_\beta = g_{\beta_c}$ so $f(u,\beta) = f(u^{\frac{\beta_c}{\beta}},\beta_c)$.

β dependence of M_β and stable subordinators

- Let $(X^\alpha(t))_{t\geq 0}$ be the non-negative increasing Levy process with Laplace transform
$$E(e^{-uX^\alpha(t)})=e^{-tu^\alpha},$$
where $\alpha\in(0,1)$ (this is a stable subordinator of index α).
- Let $\beta>\beta_c$ and take $X^{\frac{\beta_c}{\beta}}$ independent of M_{β_c}. We then have
$$\begin{aligned}E\left(e^{-uX^{\frac{\beta_c}{\beta}}(M_{\beta_c})}\right)&=E\left(E\left(e^{-uX^{\frac{\beta_c}{\beta}}(M_{\beta_c})}\middle| M_{\beta_c}\right)\right)\\&=E\left(e^{-u^{\frac{\beta_c}{\beta}}M_{\beta_c}}\right)\\&=f(u^{\frac{\beta_c}{\beta}},\beta_c)=f(u,\beta).\end{aligned}$$
- Thus $X^{\frac{\beta_c}{\beta}}(M_{\beta_c})\stackrel{d}{=}M_\beta$.

Convergence of the measures

- Barral, Rhodes and Vargas noticed that the convergence results of Aidekon and Shi and Madaule imply that if one normalizes the measures $\mu_{n,\beta}$ in the same way as the total mass $Z_{n,\beta}$, the measures converge in distribution also for $\beta > \beta_c$ and in probability for $\beta = \beta_c$.
- The basic idea behind their argument is that the distributional recursion for the total mass implies that the masses of all collections of dyadic intervals converge jointly in distribution.

Convergence of the measures

- The β-dependence of the total mass in terms of the stable subordinators extends to the measures: let ν_β be the distributional weak limit of $e^{-\beta m_{\beta,n}}\mu_{\beta,n}$ for $\beta > \beta_c$. Then

$$\nu_\beta(dx) \stackrel{d}{=} X^{\frac{\beta_c}{\beta}}(\nu_{\beta_c}(dx))$$

  This means that ν_β is given by the derivative (in the sense of distributions) of the function $x \mapsto X^{\frac{\beta_c}{\beta}}(\nu_{\beta_c}([0,x]))$.
- Also the Gibbs measures ($\mu_{\beta,n}$ normalized to have unit mass) converge and have a similar representation.

Atomic structure of the limit measures

- For $\alpha \in (0,1)$ the processes X^α are pure jump processes. This implies that for $\beta > \beta_c$, the measures ν_β are purely atomic which is consistent with the numerical results by Carpentier and Le Doussal.
- For $\beta = \beta_c$, it turns out that the measure ν_{β_c} does not have atoms (Barral, Kupiainen, Nikula, Saksman and Webb based on a result by Buraczewski).

Higher dimensions

- The hierarchical construction can be used to construct measures in higher dimensions as well (in dimension d, the branching order is 2^d)
- The proof for the convergence of the total mass and the measures themselves goes through in an identical manner.
- Replace $\log 2$ by $\log 2^d$ - $\beta_c = \sqrt{2d \log 2}$ etc.
- Barral, Rhodes and Vargas have a representation of the measures in terms of a Poisson point process which extends to any dimension.

Extreme value statistics - the Gumbel distribution

- Let (X_i) be i.i.d random variables and $M_N = \max_{i \leq N} X_i$ with distribution λ.
- Assume that there exists deterministic sequences (a_n) and (b_n) and a function $G(x)$ such that $P(a_N(M_N - b_N) \leq x) \to G(x)$.
- For a large class of distributions λ, G is given by the Gumbel distribution: $G(x) = \exp(-e^{-a(x-b)})$.
- $1 - G(x) \sim e^{-ax}$
- For a large class of (weakly) correlated random variables (X_i), the limit is also the Gumbel distribution (universality).

Extreme value statistics - beyond Gumbel

- As $\beta \to \infty$, only the 'minimum energy configuration' should contribute.
- Define $W_n(x) = \prod_{k=1}^{2^n} \mathbf{1}(x + X_{k,n} > 0)$ and $F_n(x) = E(W_n(x)) = P(\min_k X_{n,k} > -x)$.
- The self-similarity of the branching random walk implies that F_n satisfies the same recursion as $G_{n,\beta}$ with Heaviside initial data.
- Thus $F_n = G_{n,\infty}$ and by symmetry $P(\max_k X_{n,k} \leq x + m_{n,\infty}) \to g_{\beta_c}(x)$.
- $1 - g_{\beta_c}(x) \sim xe^{-\beta_c x}$ so our family of random variables $\{X_{n,k}\}$ is not in the Gumbel universality class.
- This tail behavior has been conjectured to be a universal property of logarithmically correlated models (Carpentier and Le Doussal).

Extreme value statistics - the GFF

- Fyodorov and Bouchaud: a heuristic derivation for the full distribution function for a (translation invariant) logarithmically correlated REM on the circle.
- Can such a result be proven for the 2-dimensional GFF?
- Bramson and Zeitouni; Zeitouni and Ding: If M_N is the maximum of the discrete 2-dimensional GFF on a $N \times N$ square and $m_N = E(M_N) = 2\sqrt{\frac{2}{\pi}}(\log N - \frac{3}{8}\log\log N) + \mathcal{O}(1)$, then $M_N - m_N$ is tight and for any $\epsilon > 0$ there is a C_ϵ so that for all $\lambda \in [0, \sqrt{\log N})$

$$C_\epsilon^{-1} e^{-(\sqrt{2\pi}+\epsilon)\lambda} \leq P(M_N > m_N + \lambda) \leq C_\epsilon e^{-(\sqrt{2\pi}-\epsilon)\lambda}.$$

- Note that if we consider the BRW in two dimensions (replace $\log 2$ by $\log 4$ in $m_{\beta,n}$) in the low temperature case and set $N = 2^n$, we get $m_{2^n} = \sqrt{\frac{2\log 2}{\pi}} m_{n,\beta}$.

Replica overlap

- What is known about the Gibbs measures for translation invariant logarithmically correlated models?
- Let ϕ_N be a translation invariant logarithmically correlated Gaussian field on $[0,1]$ discretized to scale N^{-1}.
- Let $\mu_{\beta,N}$ be the corresponding Gibbs measure and $q_N(x,y) = -\frac{\log ||x-y||}{\log N}$ (the normalized covariance).
- Define the replica overlap:

$$f_\beta(q) = \lim_{N\to\infty} E\left(\sum_{x,y\in X_N} \mu_{\beta,N}(x)\mu_{\beta,N}(y)\mathbf{1}(q_N(x,y)\leq q)\right)$$

- Arguin and Kistler: (for a specific ϕ_N) $\beta_c < \beta$ and $q \in [0,1)$, $f_\beta(q) = \frac{\beta_c}{\beta}$ (Derrida and Spohn; Bovier and Kurkova: this holds for the BRW as well).

Poisson-Dirichlet statistics

- Let $\alpha \in (0,1)$ and $\eta = (\eta_i)$ be a Poisson point process on $(0,\infty)$ with intensity $s^{-\alpha-1}ds$. Let $\xi = (\xi_i)$ be the sequence $\left(\frac{\eta_i}{\sum_j \eta_j}\right)$ reordered into a decreasing order. ξ is called a Poisson-Dirichlet process of parameter α.
- Arguin and Kistler: (again for a specific ϕ_N) Let $\beta_c < \beta$, ξ be a Poisson-Dirichlet process of parameter $\frac{\beta_c}{\beta}$ and $F : [0,1]^{\frac{m(m-1)}{2}} \to \mathbb{R}$ be continuous. Then

$$\lim_{n\to\infty} E\left(\sum_{x_1,\dots,x_m} \mu_{\beta,n}(x_1)\cdots\mu_{\beta,n}(x_m)F(\{q_n(x_i,x_j)\})\right)$$
$$= E\left(\sum_{k_1,\dots,k_m} \xi_{k_1}\cdots\xi_{k_s}F(\{\delta_{k_i,k_j}\})\right).$$

- This was proven for the BRW by Bovier and Kurkova.

Interpretation

- What this result says is that the mass of the low temperature Gibbs measure is distributed into clusters and the masses of these clusters form a Poisson-Dirichlet process.
- While this result is not strong enough to prove convergence of the measure or that if the limit measure exists, it is atomic, the result hints in that direction.
- This result does lose some of the structure of the model - even in the uncorrelated REM, one has the Poisson-Dirichlet statistics.

Conclusions

- For an approximation of the GFF, we have seen that the low temperature 'quantum area measures' exist and are purely atomic. We also have seen 'freezing' at low temperatures.
- Numerics and heuristic arguments suggest that this should be the case for the GFF as well.
- Rigorous results concerning extreme value statistics and the replica overlap also point in this direction.
- There are several interesting open questions to study for the GFF:
  - Exact asymptotics of the extreme value statistics.
  - Is there something similar to the distributional recursion? (Physicists in the spin glass community believe that in the limit, there should be an ultrametric structure).
  - Can one show convergence of the low temperature measure with the BRW-normalization?
  - Does the limiting measure have a representation in terms of the stable subordinator?

Heuristic arguments for the form of $m_{\beta,n}$

- Let $U_{n,\beta} = 1 - G_{n,\beta}$ and iterate the recursion for $G_{n,\beta}$. One gets a 'discrete Feynman-Kac' formula:

$$U_{n,\beta}(x) = \int \frac{e^{-\frac{(x-y)^2}{2n}}}{\sqrt{2\pi n}} U_{0,\beta}(y) E_n^{x,y} \left(e^{\sum_{m=1}^{n} k_{n-m}(Y_m)} \right) dy.$$

- Here Y is a discrete time Brownian bridge, $E_n^{x,y}$ is expectation with respect to the discrete time Brownian bridge from x to y over $\{0, ..., n\}$ and

$$k_m(x) = \log(1 + G_{m,\beta}(x)).$$

- The technical part of the proof goes into showing that if Y below the line connecting $x + m_{\beta,n}$ and y, the Feynman-Kac exponential decays very quickly. Above this line (for large x)

$$U_{n,\beta}(x+m_{n,\beta}) \approx \int \frac{e^{-\frac{(x+m_{n,\beta}-y)^2}{2n}}}{\sqrt{2\pi n}} U_{0,\beta}(y) 2^n P_n^{x,y}(Y_m > 0 \, \forall m) dy.$$

Heuristic arguments for the form of $m_{\beta,n}$

- By the Gambler's ruin estimate, $P_n^{x,y}(Y_m > 0\ \forall m) \sim \frac{xy}{n}$ for large x and y.
- Using $U_{0,\beta}(y) \sim e^{-\beta y}$ and $\frac{m_{n,\beta}}{n} \approx \beta_c$ (this follows from simpler estimates), we have

$$U_{n,\beta}(x + m_{n,\beta}) \approx 2^n e^{-\frac{m_{n,\beta}^2}{2n}} x e^{-\beta_c x} n^{-\frac{3}{2}} \int e^{-\frac{y^2}{2n}} e^{\beta_c y} y U_{0,\beta}(y) dy.$$

- If $\beta > \beta_c$, the integral converges as $n \to \infty$ and $m_{n,\beta}$ is determined by the condition that $2^n e^{-\frac{m_{n,\beta}^2}{2n}} n^{-\frac{3}{2}}$ must be bounded.
- For $\beta = \beta_c$, the integral behaves like n so $m_{n,\beta}$ is determined by the condition that $2^n e^{-\frac{m_{n,\beta}^2}{2n}} n^{-\frac{1}{2}}$ must be bounded.