Elliptic Bethe Ansatz for itinerant fermions on a chain

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Outline

- SuperSymmetry
- A SUSY lattice model
- Bethe Ansatz: what could go wrong?

A set of operators Q_i with $1 \le i \le \mathcal{N}$.
Satisfying $\{Q_i, Q_j\} = H\delta_{i,j}$.

We consider the case $\mathcal{N} = 2$, and define $Q = Q_1 + \mathrm{i}Q_2$.
Then it follows

$$Q^2 = 0$$

and we take

$$H = \{Q, Q^\dagger\} \qquad [F, Q] = Q$$

interpreting H as the Hamiltonian, and F as a fermion number.

This minimal structure has already many consequences

- $$\langle\phi|H|\phi\rangle = \|\, Q|\phi\rangle\,\|^2 + \|\, Q^\dagger|\phi\rangle\,\|^2 \ge 0$$
  only non-negative eigenvalues.
- $$H|\phi\rangle = 0 \quad \Longleftrightarrow \quad Q|\phi\rangle = Q^\dagger|\phi\rangle = 0$$

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  Proof: consider eigenstate of H

  $$E_\phi\,|\phi\rangle = H|\phi\rangle \quad \Longleftrightarrow \quad E_\phi\,|\phi\rangle = Q^\dagger Q|\phi\rangle + QQ^\dagger|\phi\rangle$$

- Take $|\phi\rangle \in Q\mathcal{H}$ and $E_\phi\,|\phi\rangle = H|\phi\rangle$.
  Then $Q^\dagger|\phi\rangle \in Q^\dagger\mathcal{H}$ has the same eigenvalue.
  non-zero energy states form doublets connected by Q and $Q^\dagger$.

- Witten Index: $\qquad W = \mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H}$
  independent of β. W is lower bound of number of E=0 states.

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Assume some lattice (any graph), and fermion operators c_i on the sites $\{c_i^\dagger, c_j\} = \delta_{i,j}$ and $\{c_i, c_j\} = 0$.
We define the Q operator as

$$Q = \sum_j c_j^\dagger p_j \qquad \text{where} \qquad p_j = \prod_{k \smile j} (1 - c_k^\dagger c_k)$$

p_j projects on the conditions that all neighbors of j are empty.

Clearly $Q^2 = 0$, because $\{c_k^\dagger p_k, c_j^\dagger p_j\} = 0$.

$$H = \{Q^\dagger, Q\} = \sum_{i \smile j} p_i \; c_i^\dagger c_j \; p_j \; + \; \sum_i p_i$$

Fairly realistic model for interacting, itinerant fermions:
Hopping, excluded neighbors, attracted second neighbors.

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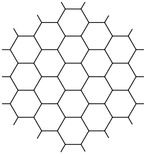
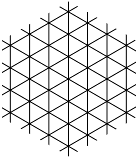
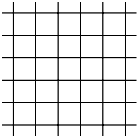






This construction is possible on any lattice.
Dimension, regular, irregular, · · ·

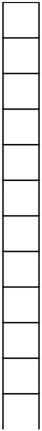
Ground state degeneracy exponential (generically)

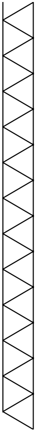
Natural way to represent dimers

Quantum criticality

Integrability

Here we focus on the simplest case:
the (periodic) chain





$$H = \sum_x p_x \; c_x^\dagger c_{x+1} \; p_{x+1} + p_{x+1} \; c_{x+1}^\dagger c_x \; p_x + p_x$$

Number of zero-energy states:

L mod 3	open	periodic
0	1	2
1	0	1
2	1	1

In the sector with $\sim L/3$ fermions.

The model can be mapped to a XXZ chain:

with appropriate Jordan-Wigner factors.

The XXZ model is integrable, and so is this fermion chain.

Many observables are surprisingly simple:
(Periodic, for $L = 3f$)

$$\langle n_x n_{x+2} \rangle = \frac{f^2 - 1}{3(4f^2 - 1)}$$

$$\langle n_x n_{x+3} \rangle = \frac{(4 - f^2 + 45f^4)}{16(4f^2 - 1)^2}$$

$$\langle n_x n_{x+4} \rangle = \frac{(-864 + 660f^2 - 2903f^4 + 1223f^6)(f^2 - 1)}{64(4f^2 - 1)^3(4f^2 - 9)}$$

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Most notable, when $L = 3f$, the ground state sector can break translational symmetry with period 3 and amplitude:

$$\langle n_{3x} - n_{3x+1} \rangle \;\propto\; \prod_{j=1}^{f} \frac{(3j-2)}{(3j-1)} \;\approx\; L^{-1/3}$$

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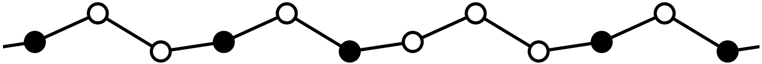

A gap opens between the ground states, linear in the system size.

First order (quantum) phase transition, sharp in a finite system.

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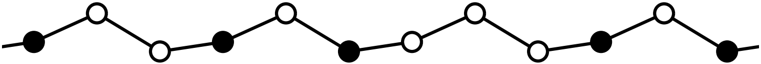

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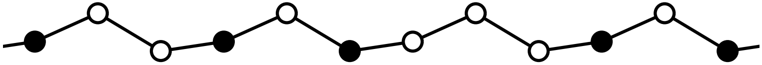

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First order (quantum) phase transition, sharp in a finite system.

But it is also possible to break the translation invariance without breaking the SuperSymmetry: Take

$$Q = \sum_x \lambda_{x \text{ mod } 3} \;\; c_x^\dagger p_x$$

The derivation of SUSY goes through unaffected by the spatial variation of λ_x.

Fendley and Hagendorf made many interesting observations about this model.

Observables for finite L are polynomial in the parameters λ_x and sometimes can be guessed from finite L results.

But is the model still integrable when λ_x varies with x?

Construction of an eigenstate of H using coordinate Bethe Ansatz.

$$H = \sum_{x=1}^{L} \lambda_x \lambda_{x+1} (p_x \; c_x^\dagger c_{x+1} \; p_{x+1} + p_{x+1} \; c_{x+1}^\dagger c_x \; p_x) + \lambda_x^2 p_x$$

General principle:

- Particles behave as plane waves as long as they are separated
- 2 particles colliding can only exchange momenta
- M-particle collisions can be described as a sequence of 2-particle collisions

These result in the following ansatz for the eigenstates of H:

$$\langle x_1, x_2, \cdots, x_f | \psi \rangle = \sum_{\pi} C_\pi \prod_j z_{\pi_j}^{x_j}$$

The plane wave assumption has to be modified:

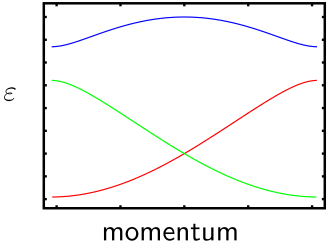
$$z^x \;\rightarrow\; A_{x \bmod 3} \; z^x$$

We normalize the λ_x by $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3$, and introduce $E = L + \varepsilon$, then the eigenvalue equation for one particle reads

$$(\varepsilon + 3)A_x = \lambda_x \sum_{y=x-1}^{x+1} \lambda_y A_y \, z^{y-x}$$

solved by the dispersion relation:

$$\varepsilon(\varepsilon + 3)^2 + \Lambda(1 - z^3)(1 - z^{-3}) = 0, \quad \text{with} \quad \Lambda = \lambda_1^2 \lambda_2^2 \lambda_3^2$$



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depends on Λ only

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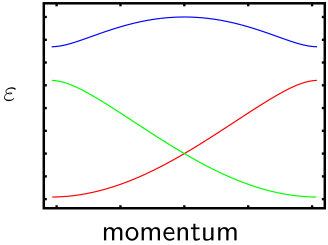

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For two particles the proposed wave function is:

$$\langle x_1, x_2|\psi\rangle = C_{12}\ A_{x_1}(z_1)z_1^{x_1}\ A_{x_2}(z_2)z_2^{x_2} + C_{21}\ A_{x_1}(z_2)z_2^{x_1}\ A_{x_2}(z_1)z_1^{x_2}$$

The eigenvalue equation for $x_2 > x_1 + 2$ leads to:

$$E = L + \varepsilon(z_1) + \varepsilon(z_2) \qquad .$$

An additional equation comes from the events that the particles are close together: $x_2 = x_1 + 2$.

This should determine the ratio C_{21}/C_{12}.

But this can happen at three inequivalent positions x_1 mod 3.

$\longrightarrow$ we have three equations for C_{21}/C_{12}.

Luckily, with the appropriate $\varepsilon(z)$ and $A_x(z)$ they agree.

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The solution:

$$S(z_1, z_2) \equiv \frac{C_{12}}{C_{21}} = -\frac{(\varepsilon(z_2)+3)\left[\varepsilon(z_2)(1-z_1^3)+\varepsilon(z_1)+3\right]}{(\varepsilon(z_1)+3)\left[\varepsilon(z_1)(1-z_2^3)+\varepsilon(z_2)+3\right]}$$

The fact that the three equations for S have a common solution, implies that the internal state of the particles is conserved during a collision.

Final test: Multiple-particle collision:

When there is a particle on site $x-2$ and x,
some exceptional terms in the eigenvalue equation must cancel:
A missing hop from $x-2$ to $x-1$, and from x to $x-1$ as well as the contribution from p_x.

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Summary BA:

1. plane waves modulated by a 3-periodic factor
2. internal state is miraculously conserved under collision
3. M-particle collisions factorizes automatically

The equations now read:

$$E = L + \sum_{j=1}^{f} \varepsilon(z_j)$$

where $\varepsilon(z)$ solves

$$\varepsilon(z)[\varepsilon(z)+3]^2 + \Lambda(1-z^3)(1-z^{-3}) = 0$$

and the z_j satisfy

$$z_j^L = \prod_{k=1}^{f} -S(z_j, z_k)$$

with

$$S(z,w) \; = \; -\frac{[\varepsilon(z)+3]\left[\varepsilon(w)(1-z^3)+\varepsilon(z)+3\right]}{[\varepsilon(w)+3]\,[\varepsilon(z)(1-w^3)+\varepsilon(w)+3]}$$

We made an attempts to resolve the multivaluedness of the dispersion relation

$$\varepsilon(z)[\varepsilon(z)+3]^2+\Lambda(1-z^3)(1-z^{-3})=0$$

seeking an analytic $\varepsilon(t)$ and $z(t)$ such that their relation is automatically satisfied.

The relation becomes more appealing by the substitution $\varepsilon = w + w^{-1} - 2$:

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What finally worked is a completely different approach:

modified difference property:

$$S(z(t_1), z(t_2)) = \frac{z(t_2)}{z(t_1)} \tilde{S}(t_2 - t_1)$$

Because this leads to the differential equation

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) \tilde{S}(t_2 - t_1) = 0$$

This leads to an ODE for $\varepsilon(z(t_1))$ from which the t_2 dependence completely separates.

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This leads to an ODE for $\varepsilon(z(t_1))$ from which the t_2 dependence completely separates.

$$\left(\frac{\partial \varepsilon}{\partial t}\right)^2 = \varepsilon\left[\varepsilon(\varepsilon+3)^2 + 4\Lambda\right]$$

This can be turned into a standard differential equation for an elliptic integral:

$$\dot{u}^2 = (1-u^2)(1-m^2u^2)$$

by repositioning the stationary points by a Möbius transformation. The results can be expressed in the Jacobi-theta functions $\theta_n(t,q)$

$$\varepsilon(t) = \text{cst.}\frac{\theta_1^2(t)}{\theta_4(t-\pi/3)\,\theta_1(t+\pi/3)}$$

$$z(t) = -\frac{\theta_1(t+\pi/3)}{\theta_1(t-\pi/3)}$$

$$S(t_1,t_2) = -\frac{z(t_1)\,\theta_1(t_1-t_2+2\pi/3)}{z(t_2)\,\theta_1(t_2-t_1+2\pi/3)}$$

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The relation between the nome q and Λ is still somewhat unpleasant

$$\Lambda = \frac{(1-\zeta^4)^2}{(1+\zeta^4/3)^3}, \qquad \zeta = \frac{\theta_1(\pi/3)}{\theta_4(\pi/3)}$$

The B.A. equations indeed turn out to be the same as those of the XYZ model.

But in this case they are constructive, i.e. there is an explicit expression for the eigenstate.

Conclusion

- Bethe Ansatz works for 3-periodic chain
- BA equations are the same as for XYZ model (modulo B.C.)
- No explicit transformation between these models (Hagendorf saturday)
- BAeq depend on Λ only $\Rightarrow$ symmetry group in $\mathcal{H}$