Hadamard's variational formula and the Gaussian free field

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Setup: Growing family of domains

Suppose $\Omega(t) \subset \mathbb{C}$ $(0 < t \leq 1)$ are bounded C^2 domains such that

(a) $\overline{\Omega(t)} \subset \Omega(t')$ for $t < t'$

(b) $\partial\Omega(t)$ has C^2 dependence on t

(c) $\Omega(t)$ has a constant finite connectivity (independent of t)

(d) the intersection $\bigcap_t \Omega(t)$ is a continuum with zero area and trivial Green function.

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(d) the intersection $\bigcap_t \Omega(t)$ is a continuum with zero area and trivial Green function.

Note: For fixed t, the boundaries $\partial\Omega(\tau)$ $(0 < \tau < t)$ almost foliate $\Omega(t)$:

$$\Omega(t) = \bigcup_{0<\tau<t} \partial\Omega(\tau) \quad \text{a.e.}$$

Moreover, we have "polar coordinates": $dA(\zeta) = \rho(\zeta)\, ds(\zeta)\, d\tau$, where

- $\rho(\zeta) =$ rate at which $\partial\Omega(\tau)$ moves at point ζ in the direction of outward normal as τ grows
- $ds =$ arc-length.

Hadamard's variational formula

Let G_t be the Dirichlet Green function of $\Omega(t)$: for each $w \in \Omega(t)$, it solves

$$\begin{cases} -\Delta G_t(\cdot, w) = \delta_w & \text{on } \Omega(t), \\ G(z, w) = 0 & \text{for } z \in \partial\Omega(t). \end{cases}$$

Hadamard's variational formula (1908):

$$G_t(z, w) = \int_0^t \left\{ \int_{\partial\Omega(\tau)} P_\tau(z, \zeta) P_\tau(w, \zeta)\, \rho(\zeta)\, ds(\zeta) \right\} d\tau$$

where

- ds = arc-length
- $P_\tau(z, \zeta) = \dfrac{\partial}{\partial \mathrm{n}(\zeta)} G_\tau(z, \zeta)$ Poisson kernel of $\Omega(\tau)$ ($= 0$ outside $\overline{\Omega(t)}$)
- $\rho(\zeta)$ = rate at which $\partial\Omega(\tau)$ moves at point ζ in the direction of outward normal as τ grows.

Reinterpretation in terms of operators

Definition

For each t, define the Hadamard operator and its adjoint:

$$Q_t f(z) = \int_{\Omega(t)} P_{\tau(\zeta)}(z,\zeta)\, f(\zeta)\, dA(\zeta)$$

$$Q_t^* f(\zeta) = 1_{\Omega(t)}(\zeta) \int_{\Omega(t)} P_{\tau(\zeta)}(z,\zeta)\, f(z)\, dA(z)$$

where $\tau(\zeta) =$ the unique τ such that $\zeta \in \partial\Omega(\tau)$.

Note: formally, Q_t maps δ_ζ onto $P_{\tau(\zeta)}(\cdot,\zeta)$.

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Proposition

(a) Q_t is an isometric isomorphism $L^2(\Omega(t)) \to H_0^1(\Omega(t))$.

(b) Q_t^* is an isometric isomorphism $H^{-1}(\Omega(t)) \to L^2(\Omega(t))$.

(c) $Q_t Q_{t'}^* = (-\Delta_{t\wedge t'})^{-1}$; in particular, $Q_t Q_t^* = (-\Delta_t)^{-1}$.

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Here:

- $H_0^1(\Omega(t)) = W_0^{1,2}(\Omega(t))$ the Sobolev space with zero boundary values under the Dirichlet inner product
$$\langle f, g \rangle_{H_0^1} = \int_{\Omega(t)} \nabla f \cdot \nabla g \, dA \qquad (dA = \text{area measure}).$$
- $H^{-1}(\Omega(t)) =$ the dual of $H_0^1(\Omega(t))$ under L^2 pairing.
- $(-\Delta_t)^{-1} =$ inverse Laplacian on $\Omega(t)$ with vanishing Dirichlet data:
$$(-\Delta_t)^{-1} f(z) = \int_{\Omega(t)} G_t(z, w)\, f(w)\, dA(w).$$

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Proof of (c):

$$Q_t Q_{t'}^* f(z) = \int\limits_{\Omega(t')} \left\{ \int\limits_{\Omega(t)} 1_{\Omega(t')}(\zeta) P_{\tau(\zeta)}(z,\zeta) P_{\tau(\zeta)}(w,\zeta)\, dA(\zeta) \right\} f(w)\, dA(w)$$

$$= \int\limits_{\Omega(t')} \underbrace{\left\{ \int_0^{t\wedge t'} \int\limits_{\partial\Omega(\tau)} P_\tau(z,\zeta) P_\tau(w,\zeta)\, \rho(\zeta)\, ds(\zeta)\, d\tau \right\}}_{= G_{t\wedge t'}(z,w) \text{ by Hadamard}} f(w)\, dA(w).$$

Gaussian free field (GFF)

$\Omega \subset \mathbb{C}$ a bounded C^2-domain.

The Gaussian free field on Ω can be (formally) defined as

$$\Psi = \sum_{j=1}^{\infty} \xi_j \psi_j$$

where (ψ_j) is an ON basis of $H_0^1(\Omega)$ and (ξ_j) are independent $N(0,1)$.

More precisely, consider the Gaussian Hilbert space consisting of the variables

$$\langle f, \Psi \rangle_{H_0^1} = \sum_{j=1}^{\infty} \xi_j \langle f, \psi_j \rangle_{H_0^1} \sim N(0, \|f\|_{H_0^1}^2), \quad f \in H_0^1(\Omega),$$

with the covariance structure

$$E\left[\langle f, \Psi \rangle_{H_0^1} \langle g, \Psi \rangle_{H_0^1}\right] = \langle f, g \rangle_{H_0^1}.$$

Gaussian free field (GFF)

Alternatively, Ψ acts on $L^2(\Omega)$ as

$$\begin{aligned} E\left[\langle f, \Psi \rangle_{L^2} \langle g, \Psi \rangle_{L^2}\right] &= \langle f, (-\Delta)^{-1} g \rangle_{L^2} \\ &= \int_\Omega \int_\Omega G(z,w)\, f(z)\, g(w)\, dA(z) dA(w) \end{aligned}$$

where G is the Green function of Ω.

GFF via Hadamard operator

Let $\Omega(t)$ be as before, and write $\Omega = \Omega(1)$ for the largest domain.

Let $\Phi =$ the white noise field on Ω:

$$\langle f, \Phi \rangle_{L^2} \sim N(0, \|f\|_{L^2}^2) \quad \text{for } f \in L^2(\Omega),$$
$$E\left[\langle f, \Phi \rangle_{L^2} \langle g, \Phi \rangle_{L^2}\right] = \langle f, g \rangle_{L^2}.$$

Define for each $0 < t \leq 1$:

$$\Psi_t = Q_t \Phi, \quad \text{i.e.} \quad \langle f, \Psi_t \rangle_{L^2} = \langle Q_t^* f, \Phi \rangle_{L^2}.$$

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Theorem

Above Ψ_t is a GFF on $\Omega(t)$.

Moreover, the process Ψ_t, $0 \leq t \leq 1$ (with $\Psi_0 \equiv 0$) has independent increments: for all $0 = t(0) < t(1) < \cdots < t(n) = 1$, the fields $\Psi_{t(j)} - \Psi_{t(j-1)}$, $j = 1, \ldots, n$, are independent.

GFF via Hadamard operator

Proof.

Let $0 < t, t' \leq 1$ and $f, g \in H^{-1}(\Omega)$. Then

$$E\big[\langle f, \Psi_t\rangle_{L^2}\langle g, \Psi_{t'}\rangle_{L^2}\big] = E\big[\langle Q_t^* f, \Phi\rangle_{L^2}\langle Q_{t'}^* g, \Phi\rangle_{L^2}\big]$$
$$= \langle Q_t^* f, Q_{t'}^* g\rangle_{L^2} = \langle f, Q_t Q_{t'}^* g\rangle_{L^2} = \langle f, (-\Delta_{t\wedge t'})^{-1} g\rangle_{L^2}$$
$$= \int_\Omega \int_\Omega G_{t\wedge t'}(z, w)\, f(z)\, g(w)\, dA(z) dA(w).$$

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Interpretation

Recall: Q_t maps δ_ζ onto $P_{\tau(\zeta)}(\cdot, \zeta)$ for $\zeta \in \Omega(t)$. This is the harmonic function on $\Omega(\tau(\zeta))$ determined by the point mass at ζ.

Thus: GFF on $\Omega(t)$ is obtained by adding (integrating) up the *harmonic fields* induced by Poisson extensions of point oscillations at points $\zeta \in \Omega(t)$.

GFF via Hadamard operator

In particular, for $0 < t < t' \leq 1$, we have the decomposition

$$\begin{aligned} \Psi_{t'} &= \Psi_t + (\Psi_{t'} - \Psi_t) \\ &= Q_t \Phi + (Q_{t'} - Q_t)\Phi \qquad \text{(independent summands)} \end{aligned}$$

where

- Ψ_t = GFF on $\Omega(t)$
- $\Psi_{t'} - \Psi_t$ = random field which is harmonic in $\Omega(t)$ and equals $\Psi_{t'}$ on $\Omega(t') \setminus \Omega(t)$.

(Markov property!)

Time-derivative of Ψ_t

Define the harmonic sweep operator from $\Omega(t)$ to $\partial\Omega(t)$ by

$$P_t^* f(\zeta) = \int_{\Omega(t)} P_t(z,\zeta)\, f(z)\, dA(z) \qquad (\zeta \in \partial\Omega(t)).$$

May write

$$\begin{aligned} E\left[\langle f, \Psi_t\rangle_{L^2} \langle g, \Psi_{t'}\rangle_{L^2}\right] &= E\left[\langle Q_t^* f, \Phi\rangle_{L^2} \langle Q_{t'}^* g, \Phi\rangle_{L^2}\right] \\ &= \langle Q_t^* f, Q_{t'}^* g\rangle_{L^2} \\ &= \int_0^{t\wedge t'} \left\{ \int_{\partial\Omega(\tau)} P_\tau^* f \; P_\tau^* g \,\rho\, ds \right\} d\tau. \end{aligned}$$

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Formally: If Ψ_t' is the time-derivative of Ψ_t, then

$$E\big[\langle f, \Psi_t'\rangle_{L^2} \langle g, \Psi_{t'}'\rangle_{L^2}\big] = \delta_0(t - t') \int_{\partial\Omega(t)} P_t^* f \; P_t^* g \, \rho \, ds.$$

Time-derivative of Ψ_t – precise statement

1. Let Ξ_t be a weighted white noise on $\partial\Omega(t)$ with weight $\rho^{1/2}$, i.e., a Gaussian random field acting on $L^2(\partial\Omega(t))$ such that

$$E\big[\langle\varphi, \Xi_t\rangle_{L^2(\partial\Omega(t))}\langle\psi, \Xi_t\rangle_{L^2(\partial\Omega(t))}\big] = \int_{\partial\Omega(t)} \varphi\psi\,\rho\,ds$$

for $\varphi, \psi \in L^2(\partial\Omega(t))$. Let $P_t\Xi_t$ be the Poisson extension of Ξ_t, i.e., a harmonic field on $\Omega(t)$ given by

$$\langle P_t\Xi_t, f\rangle_{L^2(\Omega(t))} = \langle \Xi_t, P_t^* f\rangle_{L^2(\partial\Omega(t))} \quad \text{for } f \in L^2(\Omega(t)).$$

Then the derivative of $t \mapsto \Psi_t$ in the sense of distributions has the same law as $P_t\Xi_t$.

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Then the derivative of $t \mapsto \Psi_t$ in the sense of distributions has the same law as $P_t\Xi_t$.

2. For a fixed f,

$$\langle f, \Psi_t\rangle_{L^2} \overset{d}{=} B\Big(\int_0^t \kappa(\tau)\,d\tau\Big)$$

where B is a BM and $\kappa(\tau) = \int_{\partial\Omega(\tau)} |P_\tau^* f|^2 \rho\,ds$.