Universality for isoradial bond percolation

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joint with Geoffrey Grimmett

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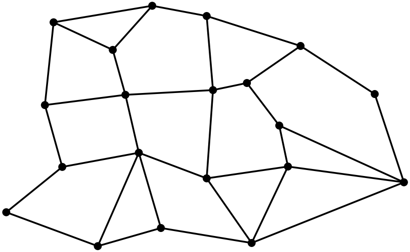
14 June 2012

Percolation



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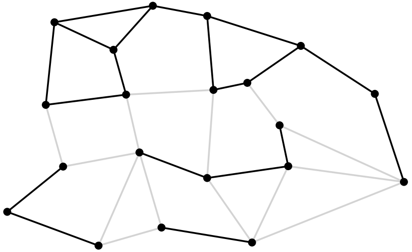

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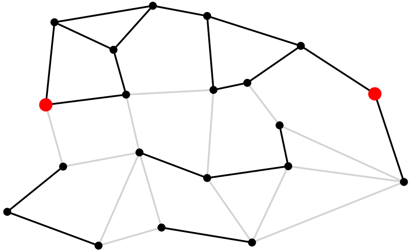

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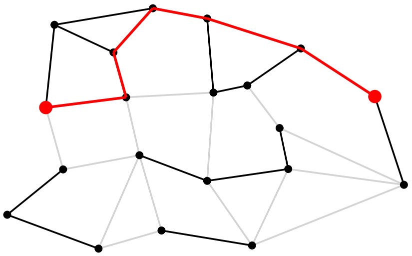

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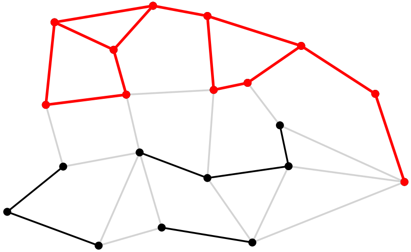

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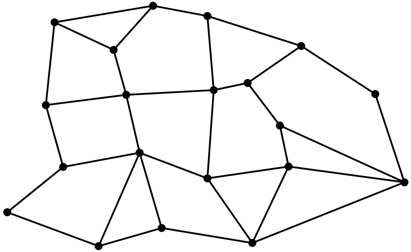

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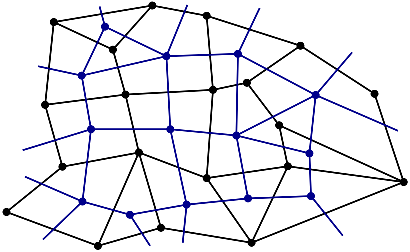

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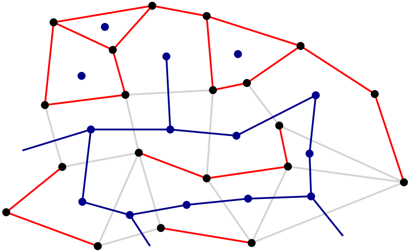

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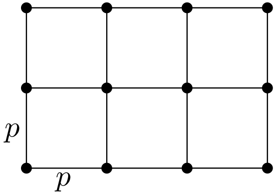
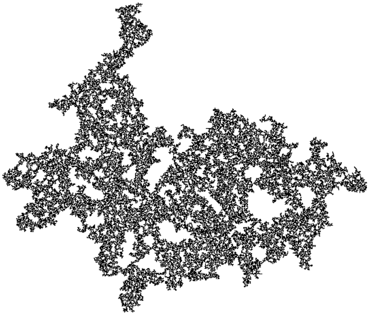
Bond Percolation on $\mathbb{Z}^2$





Phase transition:

$p < p_c$, a.s. no infinite cluster;

$p > p_c$, a.s. existence of an infinite cluster.

Criticality:

$p_c(\mathbb{Z}^2) = \frac{1}{2}$. (Kesten '80)

$\mathbb{P}_{\mathbb{Z}^2}$ - critical measure.

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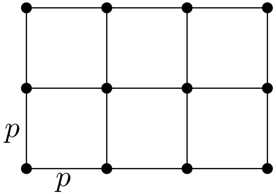

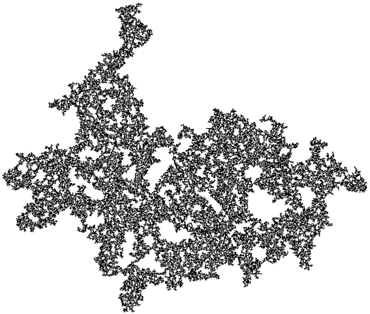

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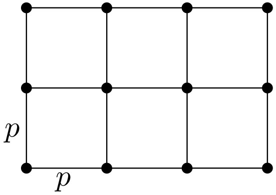

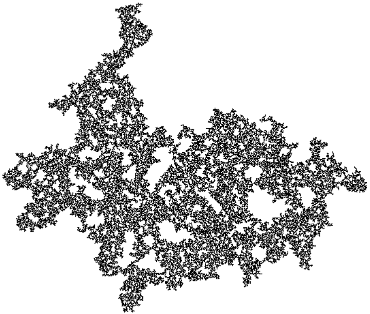

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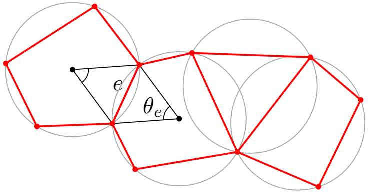
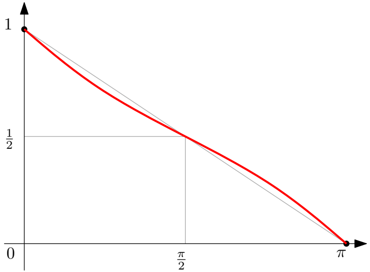
Isoradial percolation



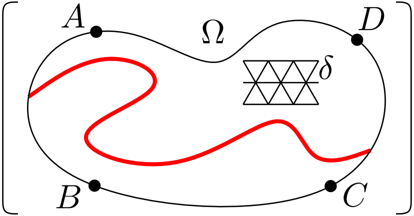
Each face of G is inscribed in a circle of radius 1.
$\mathbb{P}_G$ percolation with p_e:

$$\frac{p_e}{1-p_e} = \frac{\sin(\frac{\pi-\theta(e)}{3})}{\sin(\frac{\theta(e)}{3})}.$$



The box-crossing property (RSW)

A model satisfies the box-crossing property if for all (Ω, A, B, C, D) there exists $c(\Omega, A, B, C, D) > 0$ s. t. for all δ small enough:

$\mathbb{P}$  $\in [c, 1-c]$

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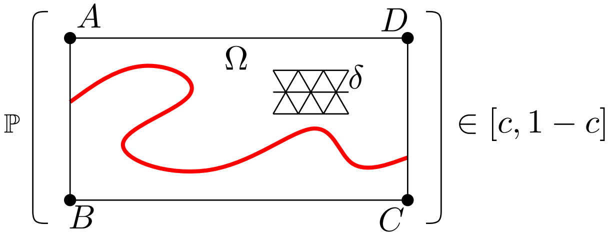

Equivalent for the primal and dual model.

Theorem

If $\mathbb{P}_{\mathbf{p}}$ satisfies the box-crossing property, then it is critical.

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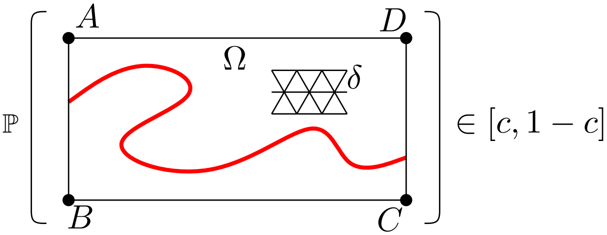

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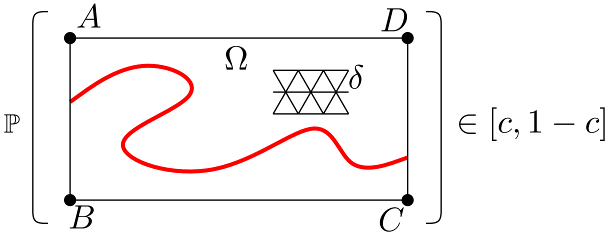

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Results I: the box-crossing property

For an isoradial graph G with the percolation measure $\mathbb{P}_G$, subject to conditions

Theorem

$\mathbb{P}_G$ *satisfies the box-crossing property.*

Corollary

$\mathbb{P}_G$ *is critical.*

- $\mathbb{P}_{\mathbf{p}}(\textit{infinite cluster}) = 0$,
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Arm exponents

For a critical percolation measure $\mathbb{P}$, as $n \to \infty$, we expect:

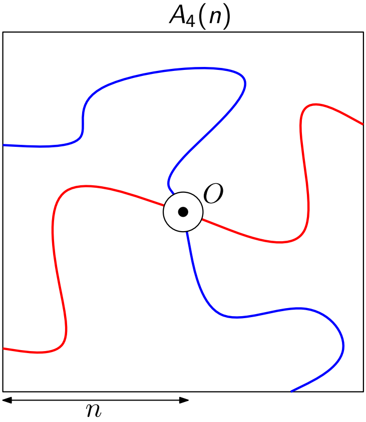
- one-arm exponent $\frac{5}{48}$:

$$\mathbb{P}(\mathrm{rad}(C_0) \geq n) = \mathbb{P}(A_1(n)) \approx n^{-\rho_1},$$

- $2j$-alternating-arms exponents $\frac{4j^2-1}{12}$:

$$\mathbb{P}[A_{2j}(n)] \approx n^{-\rho_{2j}}.$$

Moreover ρ_i **does not depend on the underlying model**.



Power-law bounds are given by the box-crossing property.

Critical exponents

For $\mathbb{P}_{\mathbf{p}}$ critical we expect:

Exponents at criticality.

Volume exponent $\delta = \frac{91}{5}$:
$\mathbb{P}_{\mathbf{p}}(|C_0| = n) \approx n^{-1-1/\delta}$.

Connectivity exponent $\eta = \frac{5}{24}$:
$\mathbb{P}_{\mathbf{p}}(0 \leftrightarrow x) \approx |x|^{-\eta}$.

Radius exponent $\rho = \frac{48}{5}$:
$\mathbb{P}_{\mathbf{p}}(\mathrm{rad}(C_0) = n) \approx n^{-1-1/\rho}$.

$(\rho = \frac{1}{\rho_1})$

Exponents near criticality.

Percolation probability $\beta = \frac{5}{36}$:
$\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0| = \infty) \approx \epsilon^{\beta}$ as $\epsilon \downarrow 0$.

Correlation length $\nu = \frac{4}{3}$:
$\xi(\mathbf{p} - \epsilon) \approx \epsilon^{-\nu}$ as $\epsilon \downarrow 0$, were
$-\frac{1}{n} \log \mathbb{P}_{\mathbf{p}-\epsilon}(\mathrm{rad}(C_0) \geq n) \to_{n\to\infty} \frac{1}{\xi(\mathbf{p}-\epsilon)}$.

Mean cluster-size $\gamma = \frac{43}{18}$:
$\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|; |C_0| < \infty) \approx |\epsilon|^{-\gamma}$ as $\epsilon \to 0$.

Gap exponent $\Delta = \frac{91}{36}$:
$\frac{\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|^{k+1}; |C_0| < \infty)}{\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|^{k}; |C_0| < \infty)} \approx |\epsilon|^{-\Delta}$. for $k \geq 1$, as $\epsilon \to 0$.

Results II: arm exponents

For an isoradial graph G with the percolation measure $\mathbb{P}_G$, subject to conditions

Theorem

For $k \in \{1, 2, 4, \ldots\}$ there exist constants $c_1, c_2 > 0$ such that:

$$c_1 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)] \leq \mathbb{P}_G[A_k(n)] \leq c_2 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)],$$

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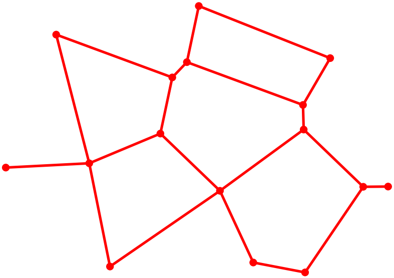
Corollary

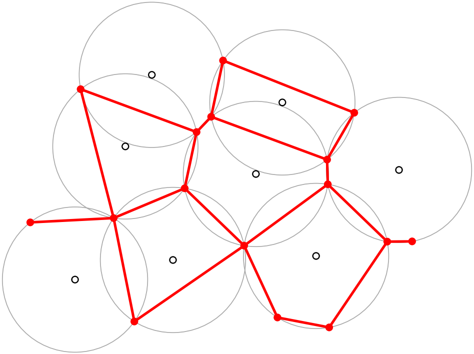
The one arm exponent and the $2j$ alternating arm exponents are universal for percolation on isoradial graphs.

Isoradial Graphs



G isoradial graph

Isoradial Graphs



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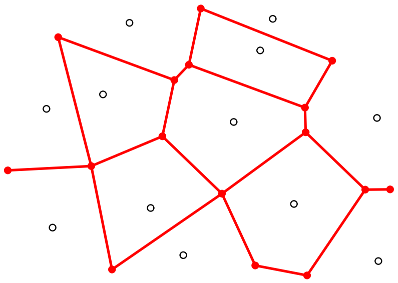

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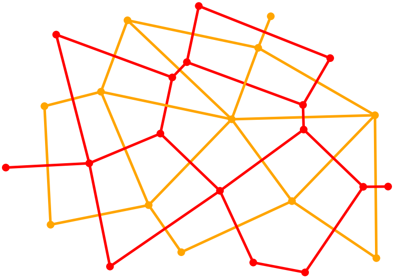

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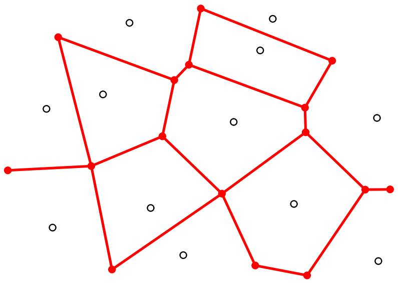

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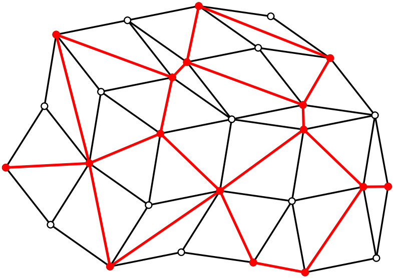

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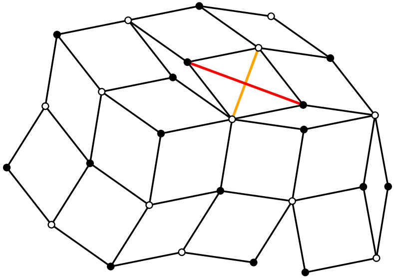

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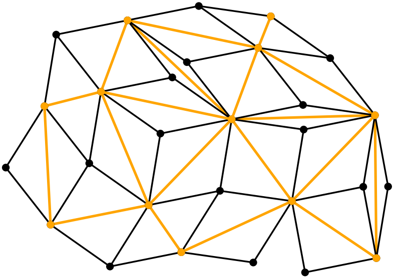

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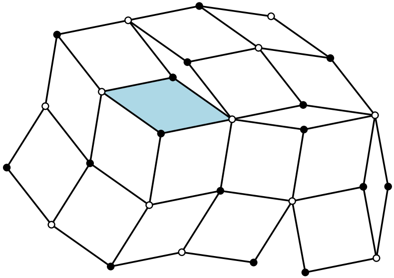

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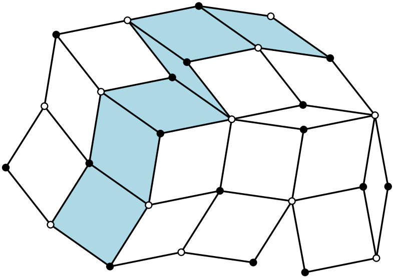

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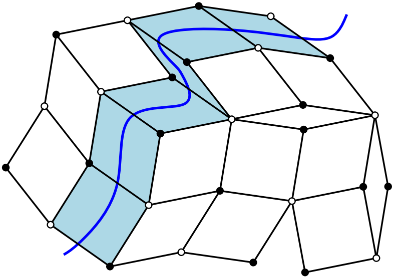

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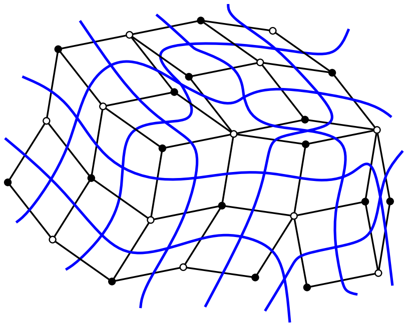

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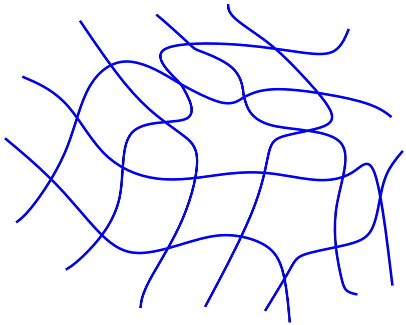

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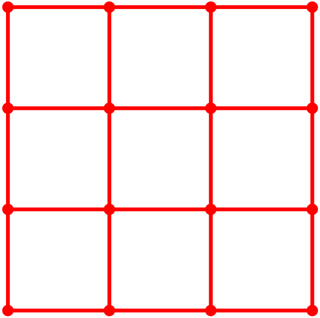

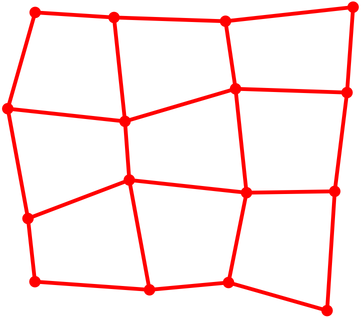

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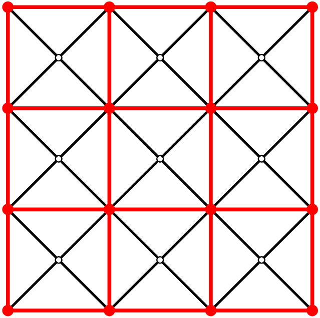

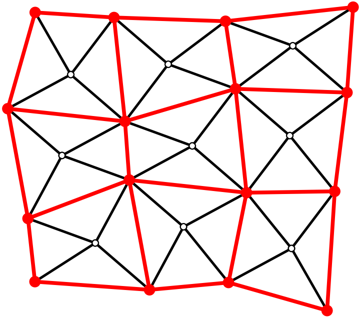

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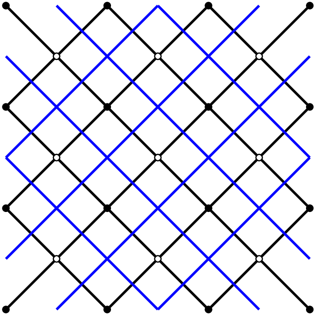

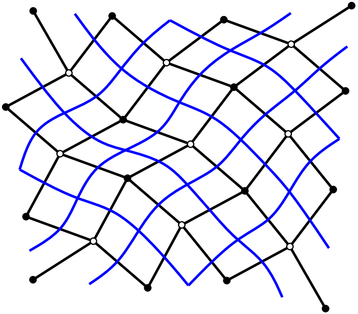

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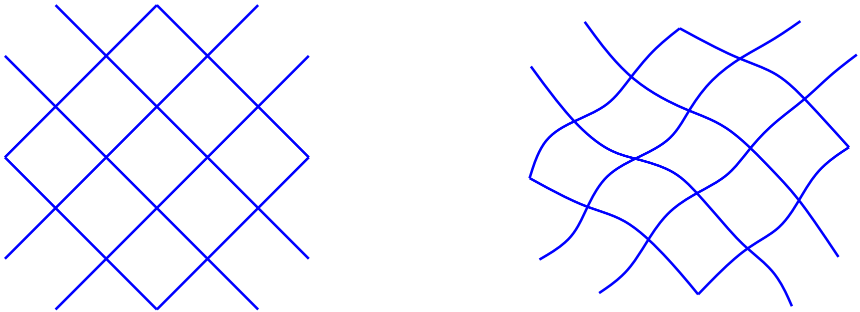

Track system similar to $\mathbb{Z}^2$ for all square lattices.

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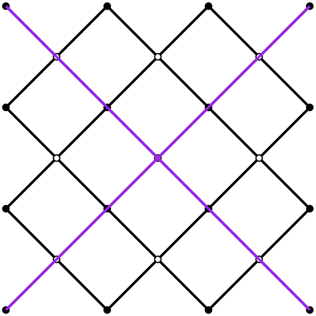

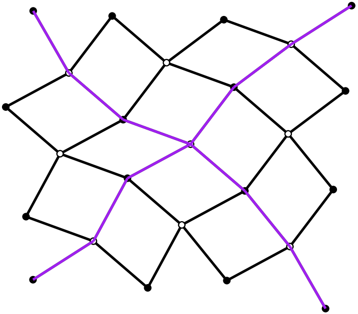

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Characterized by two sequences of transverse angles.

Conditions for isoradial graphs.

Bounded angles condition:
There exist $\epsilon_0 > 0$ such that for any edge e, $\theta_e \in [\epsilon_0, \pi - \epsilon_0]$.

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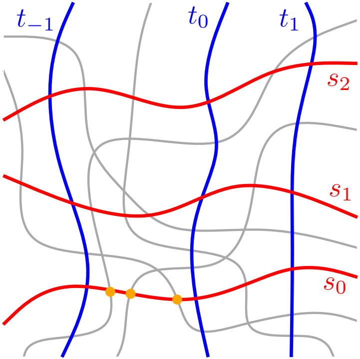
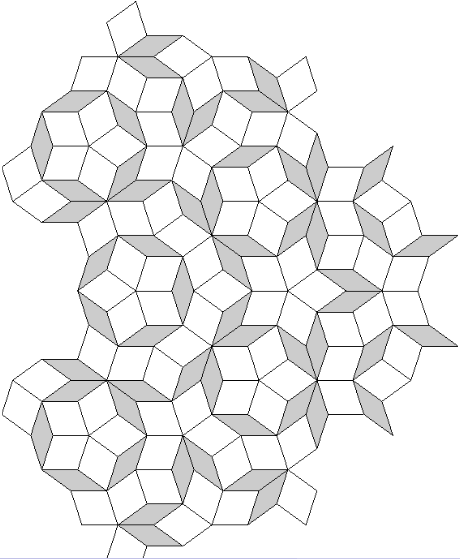


Square grid property:
Families of "parallel" tracks $(s_i)_{i\in\mathbb{Z}}$ and $(t_j)_{j\in\mathbb{Z}}$.
The number of intersections on s_i between t_j and t_{j+1} is uniformly bounded by a constant I. (same for t).

Examples: Penrose tilings and no square grid



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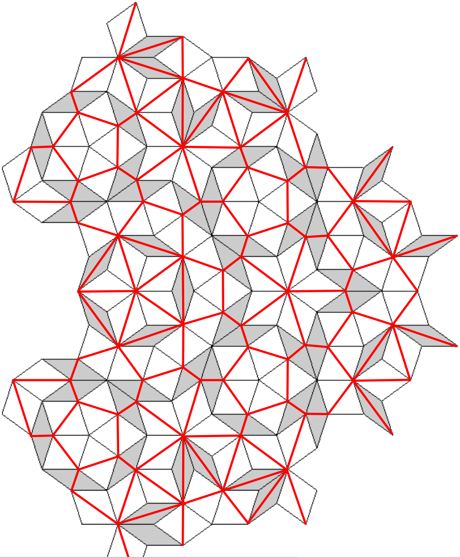

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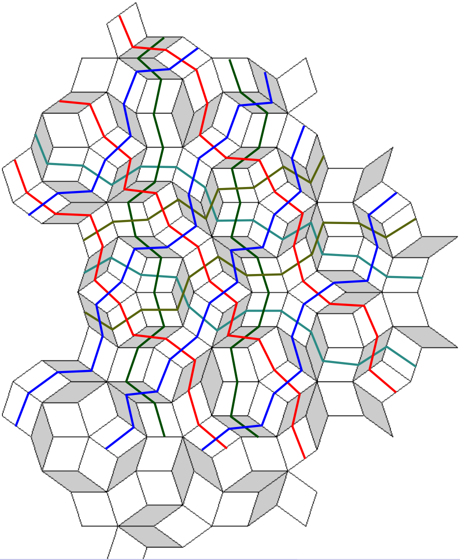

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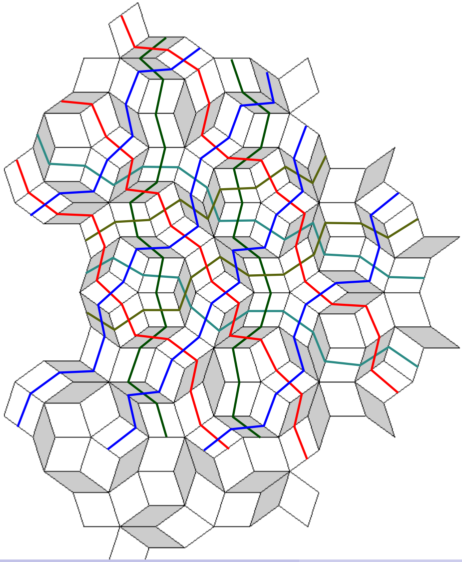

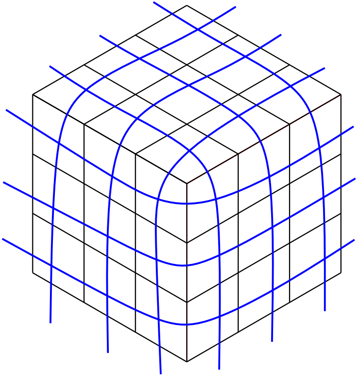

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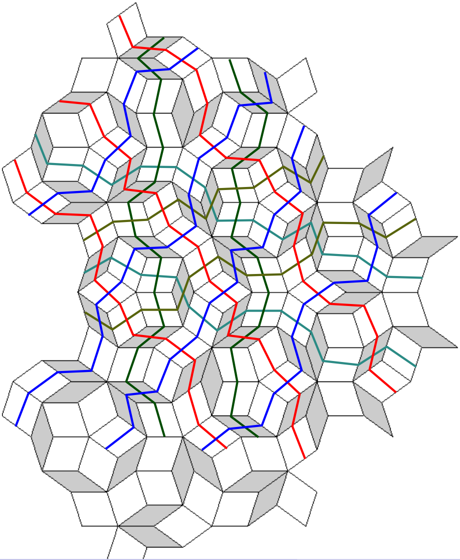

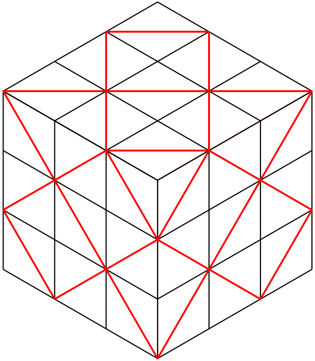

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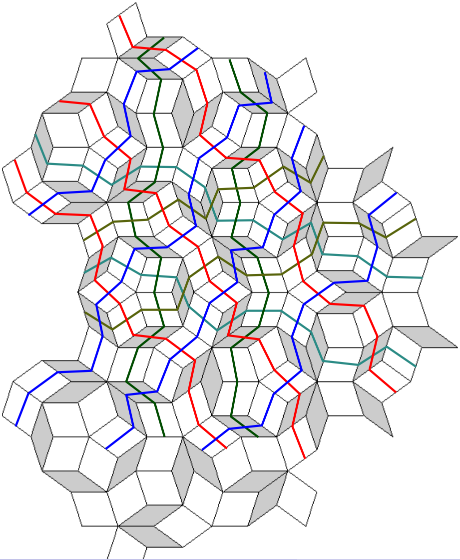
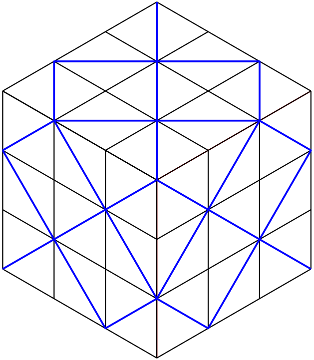
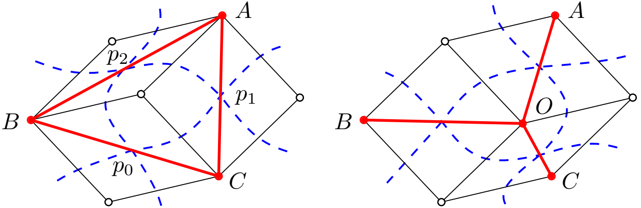




Star–triangle transformation



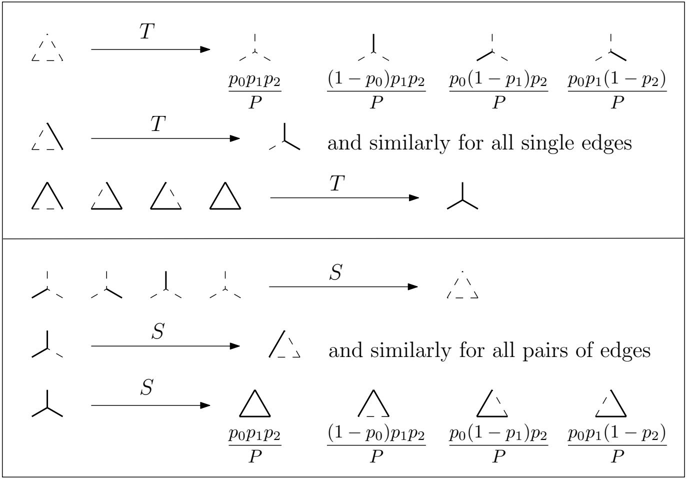
$$\kappa_{\triangle}(\mathbf{p}) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1.$$

Take ω, respectively ω', according to the measure on the left, respectively right. The families of random variables

$$\left(x \overset{\omega}{\longleftrightarrow} y : x, y = A, B, C\right), \quad \left(x \overset{\omega'}{\longleftrightarrow} y : x, y = A, B, C\right),$$

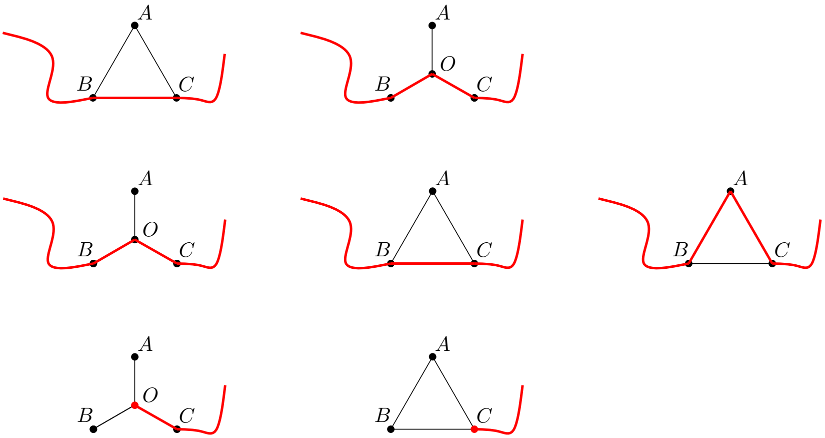
have the same joint law.

Coupling



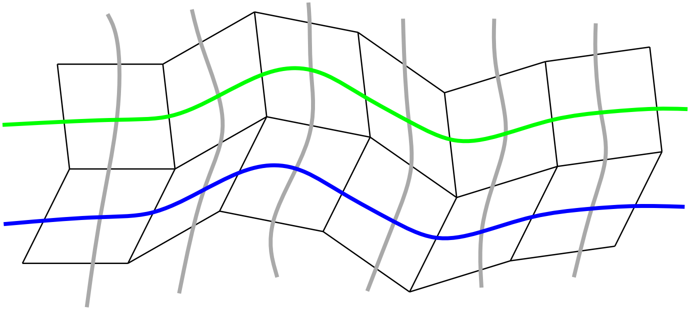
where $P = (1-p_0)(1-p_1)(1-p_2)$.

Path transformation



Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



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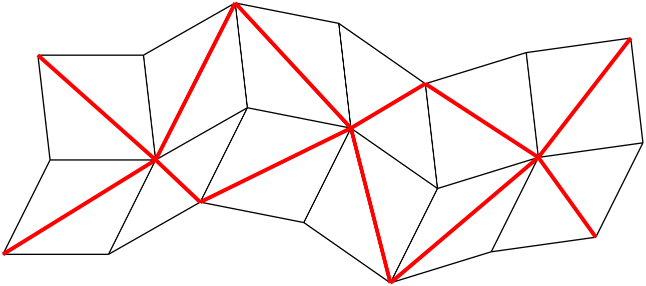

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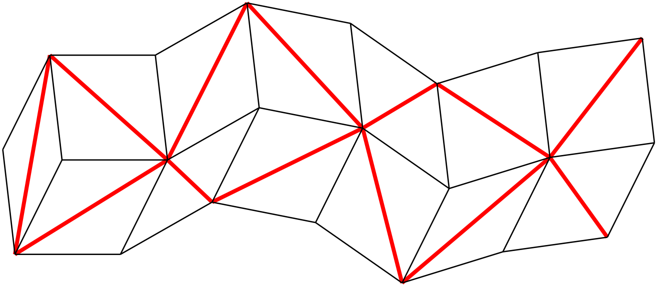

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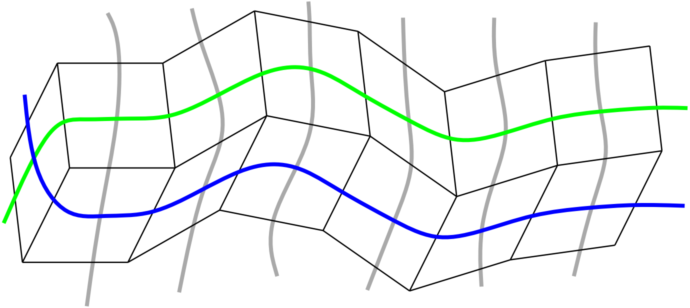

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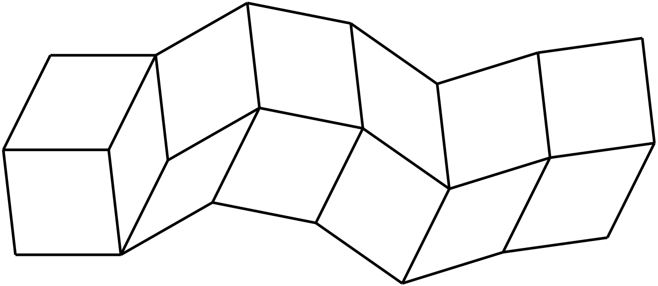

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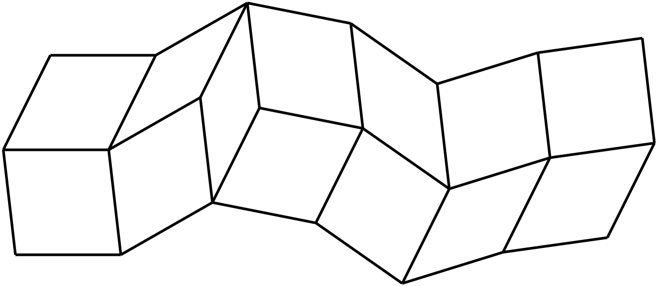

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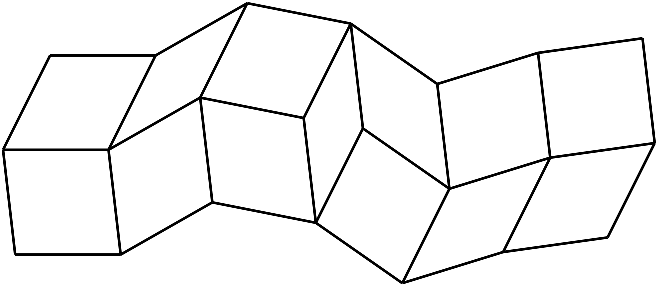

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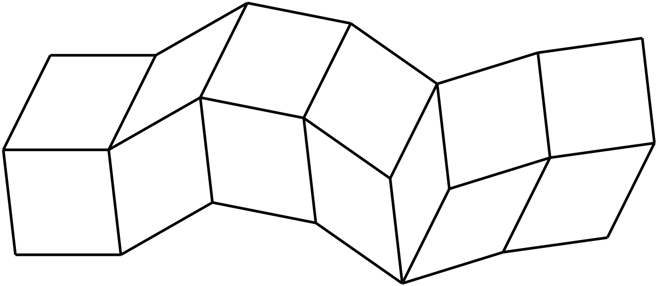

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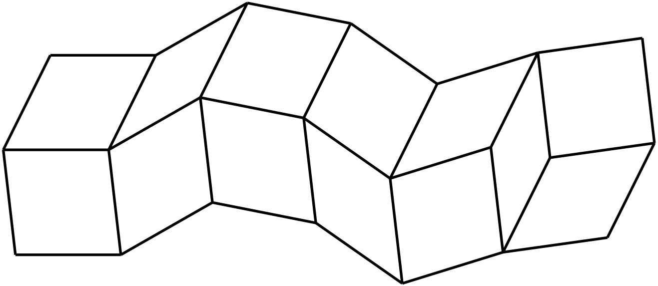

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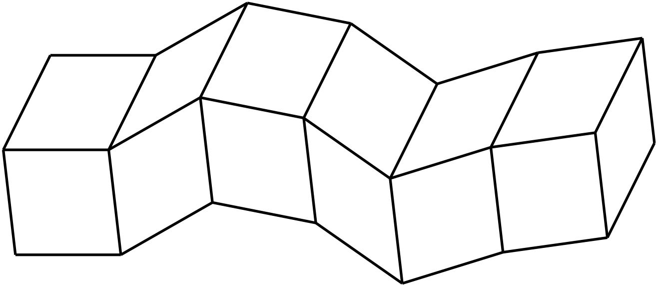

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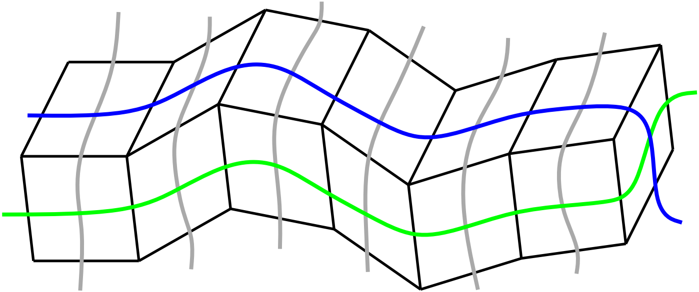

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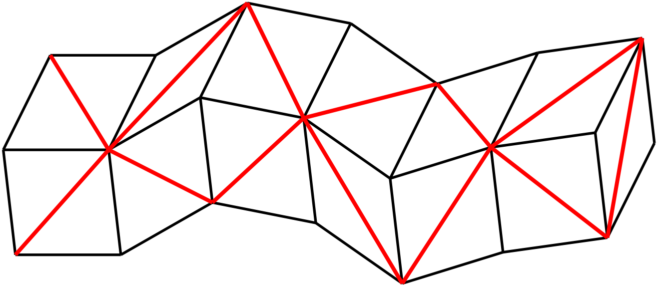

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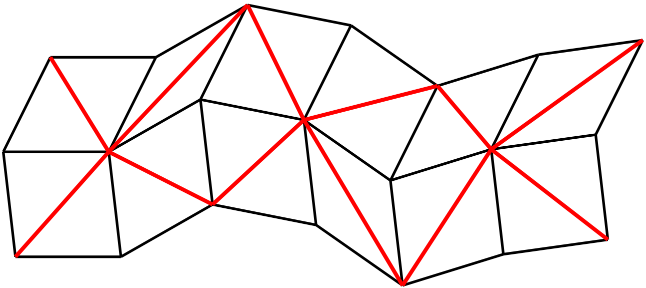

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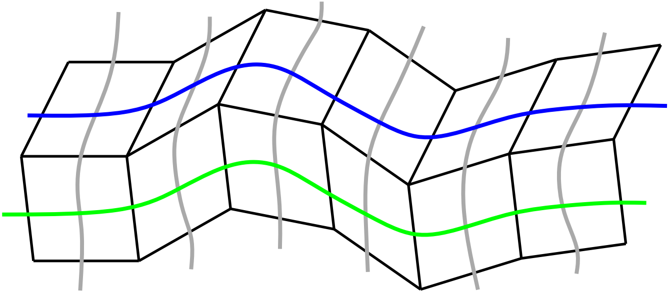
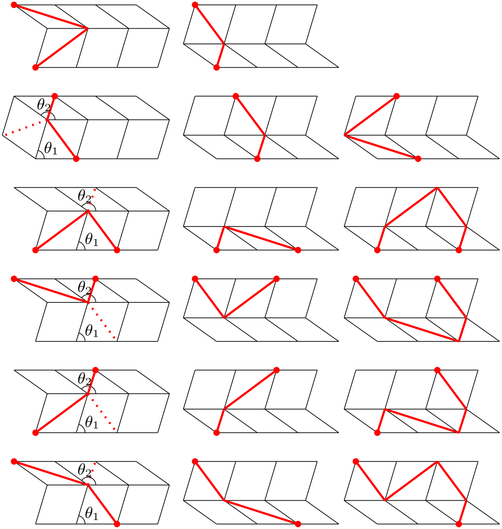


Initial configuration	Principal outcome	Secondary outcome	Probability of secondary outcome



$$\frac{p_{\pi-\theta_1}p_{\theta_2}}{p_{\theta_1}p_{\pi-\theta_2}}$$

$$\frac{p_{\pi-\theta_1}p_{\pi-\theta_2+\theta_1}}{p_{\theta_1}p_{\theta_2-\theta_1}}$$

$$\frac{p_{\theta_2}p_{\pi-\theta_2+\theta_1}}{p_{\pi-\theta_2}p_{\theta_2-\theta_1}}$$

$$\frac{p_{\theta_2}p_{\pi-\theta_2+\theta_1}}{p_{\pi-\theta_2}p_{\theta_2-\theta_1}}$$

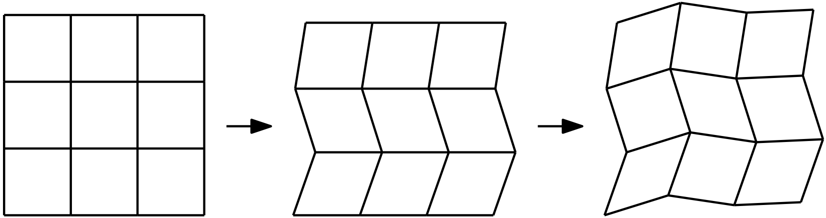
$$\frac{p_{\pi-\theta_1}p_{\pi-\theta_2+\theta_1}}{p_{\theta_1}p_{\theta_2-\theta_1}}$$

Open paths are preserved (unless the deleted edge was part of the path).

Strategy

Proposition

If two isoradial square lattices have same transverse angles for the vertical/horizontal tracks, and one has the box-crossing property, then so does the other.



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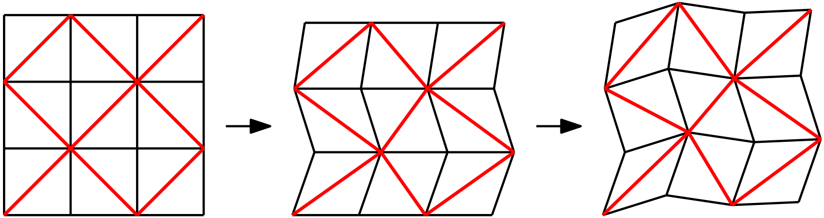
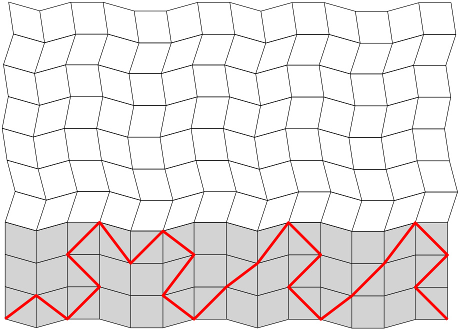


Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



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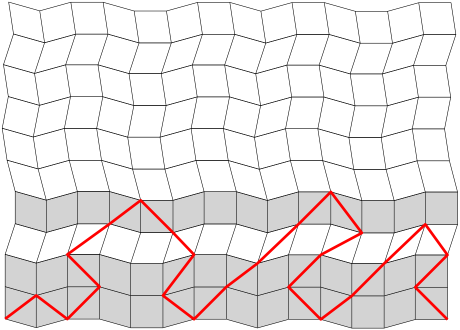

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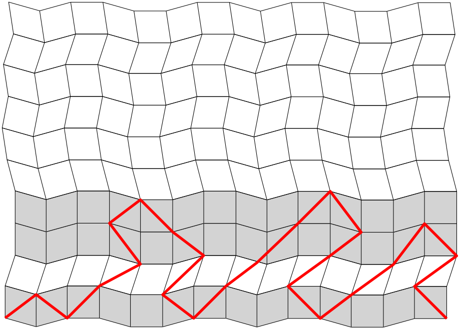

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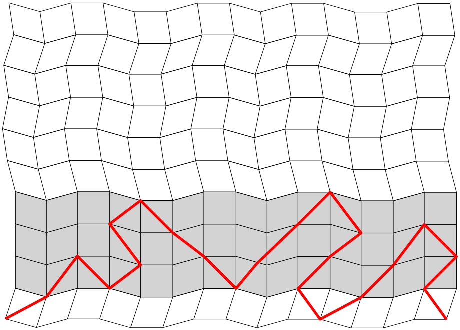

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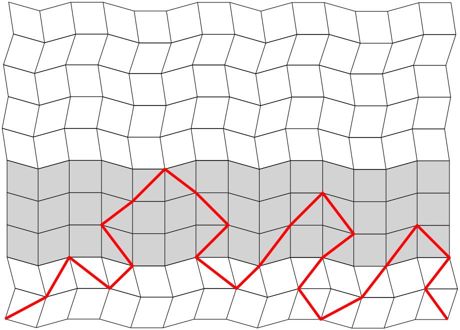

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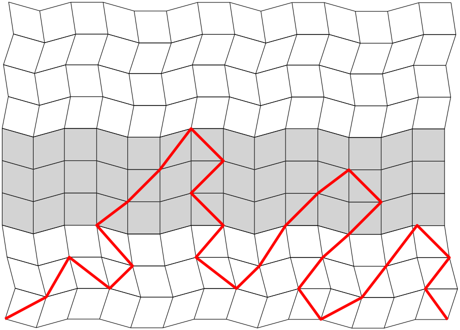

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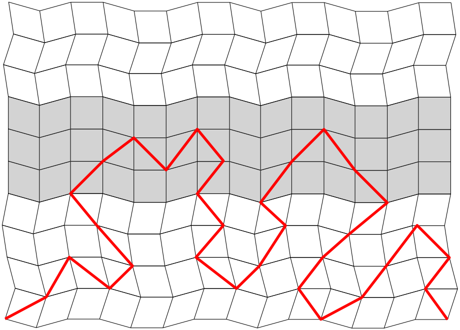

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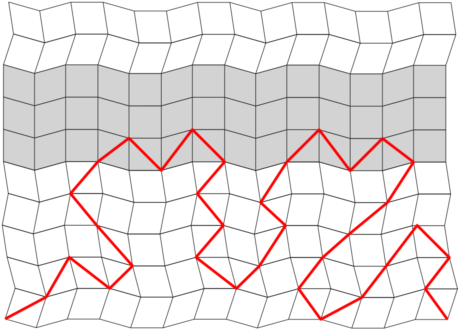

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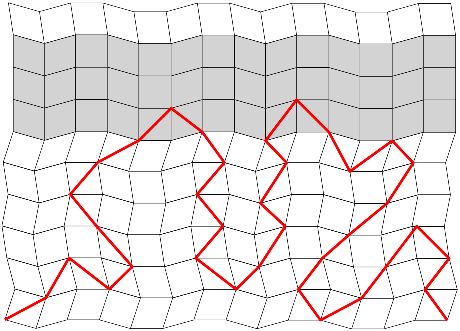

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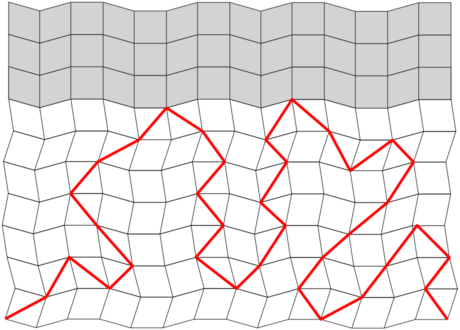

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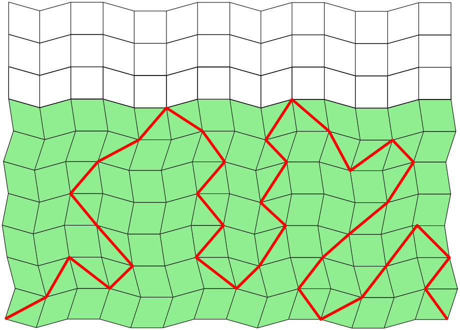
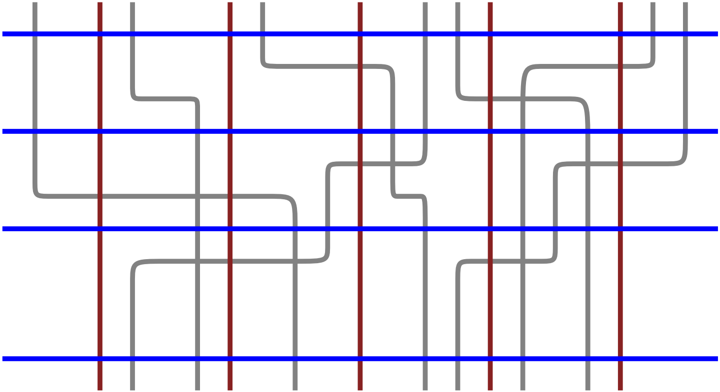


Track stacking



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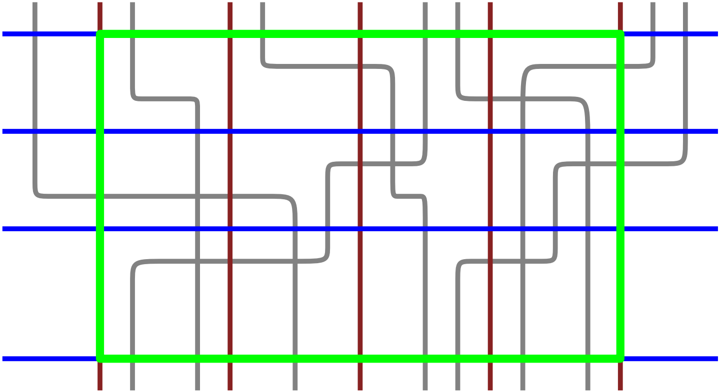

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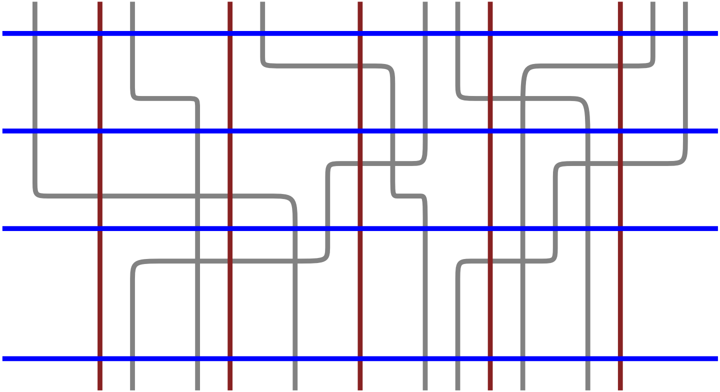

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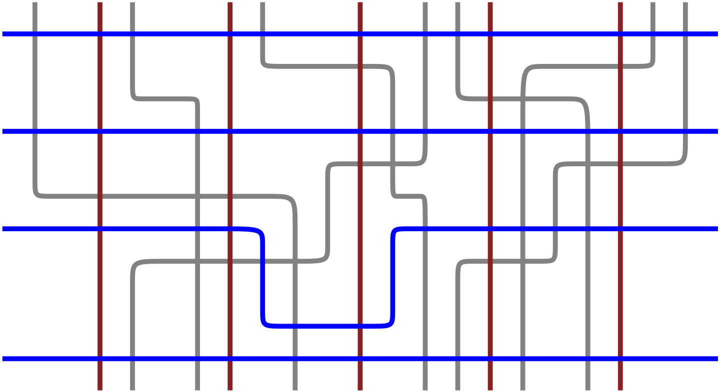

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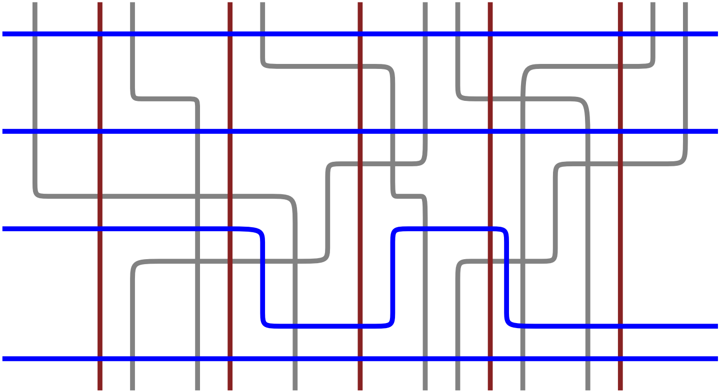

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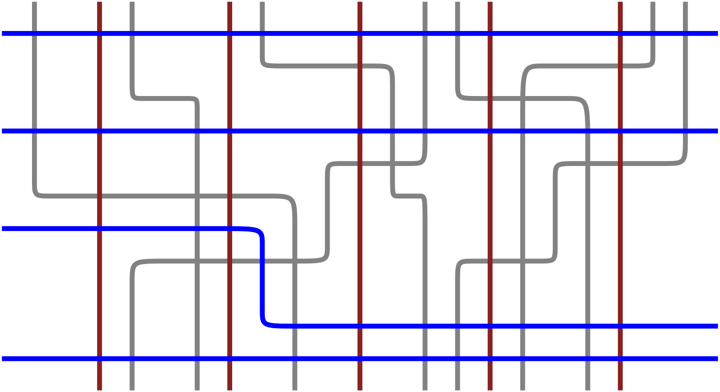

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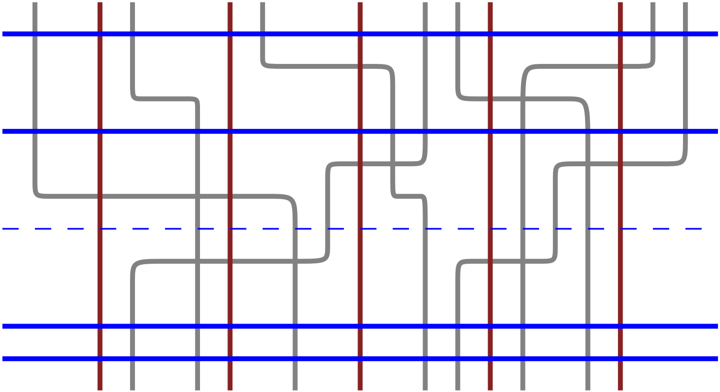

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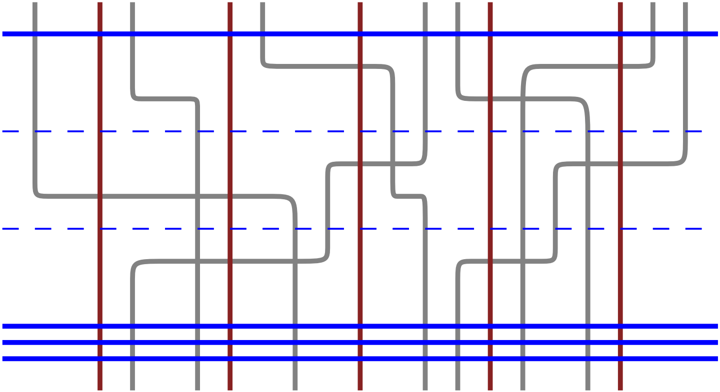

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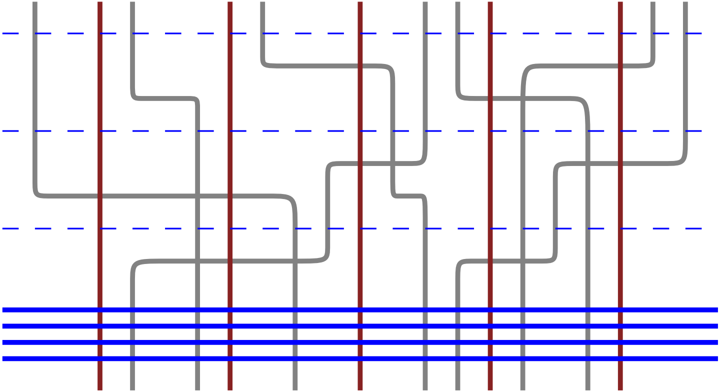

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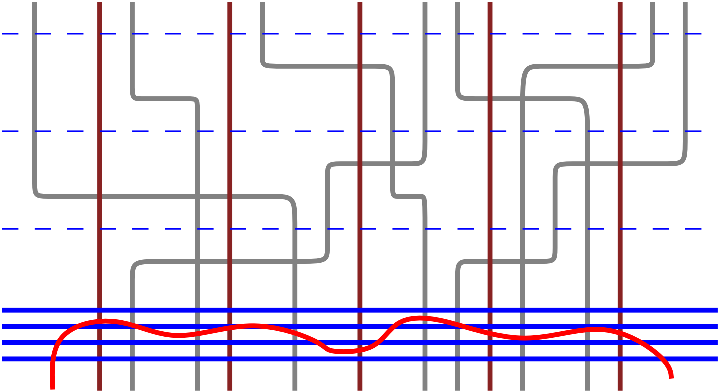

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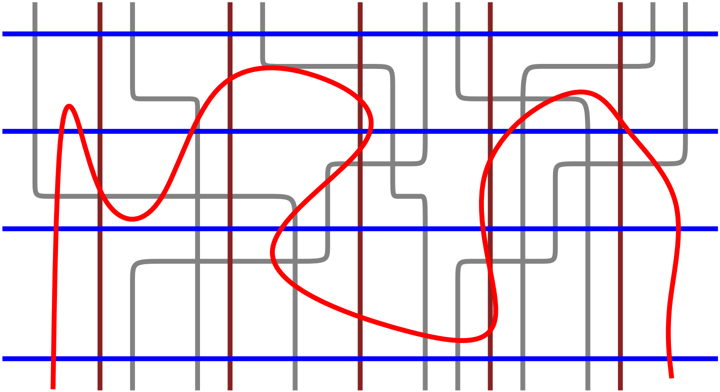

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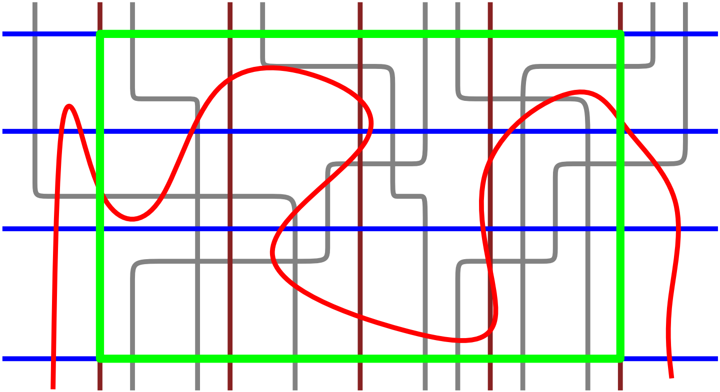

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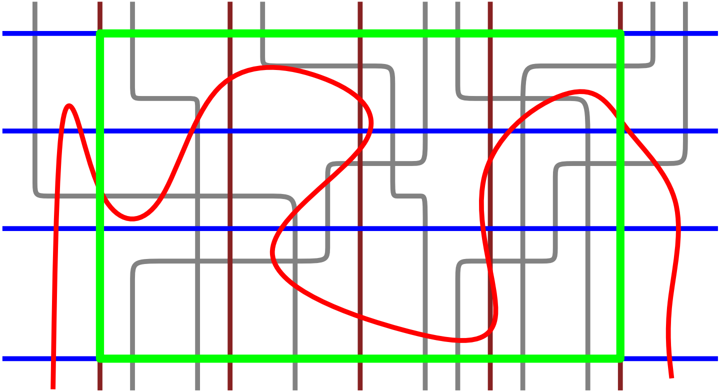

$$\mathbb{P}_{gen}(C_{\mathrm{h}}[B(\rho N, N)]) \geq \mathbb{P}_{sq}(C_{\mathrm{h}}[B(I\rho N, N)])\mathbb{P}_{sq}(C_{\mathrm{v}}[B(N, N)])^2$$

Thank you!