SOME RECENT RESULTS ON THE SCHRAMM-LOEWNER EVOLUTION (SLE)

Gregory F. Lawler
University of Chicago

SCHRAMM-LOEWNER EVOLUTION (SLE_κ)

- Measure $\mu_D(z, w)$ on paths γ connecting distinct points of a domain $D \subset \mathbb{C}$.

- The endpoints z, w can be boundary points or interior (bulk) points.

- The total mass $\Psi_D(z, w) = \|\mu_D(z, w)\|$ is called the *partition function*. It is expected to be the normalized limit of discrete partition functions.

- If $0 < \Psi_D(z,w) < \infty$, we can define a probability measure

$$\mu_D^{\#}(z,w) = \frac{\mu_D(z,w)}{\Psi_D(z,w)}.$$

- The measure will be supported on curves of a particular fractal dimension that we denote d.

$$d = \min\left\{ 1 + \frac{\kappa}{8} \, , \; 2 \right\}.$$

- How does the measure change under conformal transformation?

- How does the measure change under perturbation of the domain?

- What is the conditional distribution of the curve given some part of the curve?

- What is the probability of getting near to a point $\zeta \in D$?

We expect that the curve is parametrized by the normalized version of the length of the curve. Suppose

$$f : D \longrightarrow f(D)$$

is a conformal transformation. Since the curve has fractal dimension d, the time to traverse $f(\gamma[s,t])$ should be

$$\int_s^t |f'(\gamma(r))|^d \, dr.$$

- If μ is a measure on curves from z to w in D, let Define $f \circ \mu$ by

$$f \circ \mu[E] = \mu\{\gamma : f \circ \gamma \in E\}.$$

CONFORMAL COVARIANCE

- The conformal covariance rule is

$$f \circ \mu_D(z,w) =$$

$$|f'(z)|^{b_z}\, |f'(w)|^{b_w}\, \mu_{f(D)}(f(z),f(w)),$$

where $b_\zeta = b$ if $\zeta \in \partial D$ and $b_\zeta = \tilde{b}$ if $\zeta \in D$ where

$$b = \frac{6-\kappa}{2\kappa}, \quad \tilde{b} = \frac{b\,(\kappa-2)}{4}.$$

This assumes smoothness at boundary points.

- The probability measures are conformally *invariant*

$$f \circ \mu_D^{\#}(z,w) = \mu_{f(D)}^{\#}(f(z),f(w)).$$

The measure $\mu_D^{\#}(z,w)$ can be defined even if the boundary is not smooth.

SCHRAMM'S CONSTRUCTION

- D simply connected, $z \in \partial D$, $w \in D \cup \partial D$.

- Constructed $\mu_D^{\#}(z, w)$ as a measure on curves *modulo reparametrization*. Used a capacity parametrization.

- Assumptions were conformal invariance and *domain Markov property*: given an initial configuration $\gamma_t = \gamma[0, t]$ the distribution of the remainder of the path is

$$\mu_{D \setminus \gamma_t}^{\#}(\gamma(t), w).$$

- There is a one-parameter family of curves, indexed by κ with fractal dimension (Rohde-S, Beffara)

$$d = \min\left\{1 + \frac{\kappa}{8}, 2\right\}.$$

The curves are simple for $\kappa \leq 4$ and plane-filling for $\kappa \geq 8$.

- Reparametrized the curve so that the conformal map satisfies a stochastic differential equation. This equation is the basis for most analysis.

PARTITION FUNCTION FOR SLE_κ FOR SIMPLY CONNECTED DOMAINS

- The partition function for SLE_κ is normalized so that $\Psi_{\mathbb{H}}(0,\infty) = \Psi_{\mathbb{D}}(0,1) = \Psi_{\mathbb{H}}(0,1)$ $= \Psi_{\mathbb{C}}(0,\infty) = 1$.

- Scaling rule

  $$\Psi_D(z,w) = |f'(z)|^{b_z}\, |f'(w)|^{b_w}\, \Psi_{f(D)}(f(z), f(w)),$$

  where b_ζ is either the *boundary scaling exponent* $b = (6-\kappa)/(2\kappa)$ or the *interior scaling exponent* $\tilde{b} = b\,(\kappa-4)/2$. In particular,

  $$\Psi_{\mathbb{H}}(0,x) = |x|^{-2b}.$$

GENERALIZED RESTRICTION (L. - S. - Werner)

- The values of the scaling exponent are obtained from computations on Schramm's original process.

- Suppose simply connected $D = \mathbb{H} \setminus K$ with K compact and away from 0. Let $z \in \overline{D}$, and τ a stopping time for an SLE path such that $\gamma_\tau = \gamma(0, \tau]$ does not hit $\mathbb{H} \cap \partial D$.

- Let $\mu = \mu_{\mathbb{H}}(0,\infty), \hat{\mu} = \mu_D(0,z)$. For small τ, considered as measures on γ_τ, the two measures (on paths modulo reparametrization) are absolutely continuous with Radon-Nikodym derivative

$$\frac{d\tilde{\mu}}{d\mu}(\gamma_\tau) = \frac{\Psi_{D_\tau}(\gamma(\tau),z)}{\Psi_{\mathbb{H}_\tau}(\gamma(\tau),\infty)} \exp\left\{\frac{\mathbf{c}}{2}\Lambda_{\mathbb{H}}(\gamma_\tau, \mathbb{H}\setminus D)\right\}.$$

- Here D_τ is the connected component of $D \setminus \gamma_\tau$ containing z, and $\mathbf{c}$ is the *central charge*

$$\mathbf{c} = b(3\kappa - 8) = \frac{(6-\kappa)\,(3\kappa-8)}{2\kappa}.$$

- Although the partition functions in the ratio

$$\frac{\Psi_{D_\tau}(\gamma(\tau), z)}{\Psi_{\mathbb{H}_\tau}(\gamma(\tau), \infty)}$$

do not exist, the ratio is well defined.

- Let $g : \mathbb{H}_\tau \to \mathbb{H}$ be a conformal transformation with $g(\infty) = \infty, g'(\infty) = 1$. Then formally we can write the ratio as

$$\frac{|g'(\gamma(\tau))|^b \, |g'(z)|^{b_z} \, \Psi_{g(D_\tau)}(g(\gamma(\tau)), g(z))}{|g'(\gamma(\tau))|^b \, |g'(\infty)|^b \, \Psi_{\mathbb{H}}(g(\gamma(\tau)), \infty)}$$

- Cancelling the infinite term, and using $g'(\infty) = 1$, $\Psi_{\mathbb{H}}(x, \infty) = 1$, this gives

$$|g'(z)|^{b_z} \, \Psi_{g(D_\tau)}(g(\gamma(\tau)), g(z)).$$

- If $z = \infty$, then this term equals

$$\Psi_{g(D_\tau)}(g(\gamma(\tau)), \infty).$$

For $\kappa \leq 4$, we can let $\tau = \infty$ and write

$$\frac{d\tilde{\mu}}{d\mu}(\gamma_\tau) = \frac{\Psi_{D_\tau}(\gamma(\tau), z)}{\Psi_{\mathbb{H}_\tau}(\gamma(\tau), \infty)} \exp\left\{\frac{\mathbf{c}}{2} \Lambda_{\mathbb{H}}(\gamma, \mathbb{H} \setminus D)\right\}.$$

BROWNIAN LOOP MEASURE (L.-W.)

- Infinite (σ-finite) measure ν_D on unrooted, intersecting, loops in D.

- Characterized up to multiplicative constant by two properties:

  - **Restriction property**. If $D \subset \mathbb{C}$, the measure ν_D is obtained by restricting $\nu_{\mathbb{C}}$ to loops that lie in D.

  - **Conformal invariance**. If $f : D \to f(D)$ is a conformal transformation, then

$$\nu_{f(D)} = \nu_D$$

- $\Lambda_D(V, V')$ is the measure on the set of loops in D that intersect both V and V'.

GREEN'S FUNCTION

The Green's function $G_D(\zeta; z, w)$ for $\kappa < 8$ is the $\mu_D^{\#}(z, w)$ probability that a path goes through (near) $\zeta \in D$.

- It satisfies the scaling rule

$$G_D(\zeta; z, w) = |f'(\zeta)|^{2-d} G_{f(D)}(f(\zeta); f(z), f(w)).$$

- Characterized (up to multiplicative constant) by the fact that

$$M_t = G_{D \setminus \gamma_t}(\zeta; \gamma(t), w)$$

is a local martingale.

- (Rohde-Schramm)

$$G_{\mathbb{H}}(\zeta; 0, \infty) = [\mathrm{Im}\, \zeta]^{d-2} [\sin \arg\, \zeta]^{\frac{8}{\kappa}-1}.$$

- Let $\Upsilon_D(z)$ denote one-half times the conformal radius of z in D, that is,

$$\Upsilon_{\mathbb{H}}(z) = \mathrm{Im}(z),$$

$$\Upsilon_{f(D)}(f(z)) = |f'(z)|\, \Upsilon_D(z).$$

$$\frac{\mathrm{dist}(z, \partial D)}{2} \leq \Upsilon_D(z) \leq 2\,\mathrm{dist}(z, \partial D).$$

Theorem: As $\epsilon \downarrow 0$,

$$\mathbf{P}\{\Upsilon_{D \setminus \gamma_\infty}(\zeta) < \epsilon\} \sim c_*\, G_D(\zeta; z, w)\, \epsilon^{2-d},$$

$$c_* = 2\left[\int_0^\pi \sin^{8/\kappa} x\, dx\right]^{-1}.$$

RADIAL GREEN'S FUNCTION

Alberts-Kozdron-L.

- Write $\zeta = e^{2iz}, \quad z = x + iy$.

$$G_{\mathbb{D}}(\zeta; 1, 0) = \Phi(\zeta)\, \Upsilon_{\mathbb{D}}(\zeta)^{d-2} \left[\frac{\sinh y \, \cosh y}{|\sin z|}\right]^{\frac{8}{\kappa}-1} .$$

- The extra term $\Phi(\zeta)$ can be written as

$$\Phi(\zeta) = \mathbf{E}^* \left[g'(0)^q \right], \quad \beta = \frac{(4-\kappa)\,(\kappa-8)}{8\kappa}$$

where $\mathbf{E}^* = \mathbf{E}^*_\zeta$ is an expectation with respect to radial SLE_κ *conditioned to go through* ζ (and stopped when it reaches ζ). Here g is a conformal transformation

$$g : \mathbb{D} \longrightarrow \mathbb{D} \setminus \gamma$$

fixing the origin.

- As in chordal case, as $\epsilon \downarrow 0$,

$$\mathbf{P}\{\Upsilon_{\mathbb{D}\setminus\gamma_\infty}(\zeta) < \epsilon\} \sim c_* \, G_{\mathbb{D}}(\zeta; 1, 0) \, \epsilon^{2-d}.$$

TWO-POINT GREEN'S FUNCTION

- Consider SLE_κ in $\mathbb{H}$ and let $\Upsilon(\cdot) = \Upsilon_{\mathbb{H}\setminus\gamma_\infty}(\cdot)$.

- (L.-Werness) There exists a function $G(z,w)$ such that as $\epsilon, \delta \downarrow 0$,

  $$\mathbf{P}\{\Upsilon(z) < \epsilon, \Upsilon(w) < \delta\} \sim c_*^2 \, G(z,w) \, (\epsilon\delta)^{2-d}.$$

- Can write $G(z,w) = \hat{G}(z,w) + \hat{G}(w,z)$ where

  $$\hat{G}(z,w) = G(z) \, \mathbf{E}_z^* \left[G_{\mathbb{H}\setminus\eta}(z,w) \right].$$

  Here $\mathbf{E}_z^*$ indicates an expectation with respect to SLE conditioned to go through z and η is the path stopped at z.

- Explicit form is unknown, but (L.-W.-Rezaei)

  $$G(z,w) \asymp q^{d-2} \, [q \vee \sin \arg\, w]^{-\beta}, \quad |z| \le |w|$$

  $$q = \frac{|w-z|}{|w|}, \quad \beta = \frac{\kappa}{8} + \frac{8}{\kappa} - 2.$$

NATURAL PARAMETRIZATION (NATURAL LENGTH)

- Would like to parametrize SLE_κ such that it is the normalized limit of the length.

- Let D be a bounded simply connected domain with distinct boundary points z, w. Let γ be an SLE_κ from z to w (with any increasing continuous parametrization).

- Let Θ_t denote the "natural length" of $\gamma[0, t]$. (The curve is in the natural parametrization if $\Theta_t = t$.)

- Expect (up to multiplicative constant) that

$$\Theta_\infty = \int_D G_D(\zeta; z, w)\, dA(\zeta).$$

$$\mathbf{E}\left[\Theta_\infty \mid \gamma_t\right] = \Theta_t + \mathbf{E}\left[\Theta_\infty - \Theta_t \mid \gamma_t\right].$$

$$\mathbf{E}\left[\Theta_\infty - \Theta_t \mid \gamma_t\right] = \int G_{D\setminus\gamma_t}(\zeta;\gamma(t),z)\,dA(z).$$

- Θ_t is *defined* to the unique increasing process such that

$$\Theta_t + \int G_{D\setminus\gamma_t}(\zeta;\gamma(t),z)\,dA(z)$$

  is a martingale. (Doob-Meyer decomposition)

- Second moment estimates needed to guarantee that this is nontrivial. Similar to construction and estimates of two-point Green's function.

- (L-Sheffield) The natural length exists and is nontrivial for $\kappa < 5.0\cdots$. If γ has the (usual) capacity parametrization, then $t \mapsto \Theta_t$ is Hölder continuous.

- (L-Zhou) The natural length exists for $\kappa < 8$. The proof uses the two-point Green's function.

- (L.-Rezaei) A new proof for $\kappa < 8$ that shows that $t \mapsto \Theta_t$ is Hölder continuous. Moveover, the "length" of a path is independent of the domain. The proof uses the Hölder continuity of γ in the capacity parametrization.

INDEPENDENCE OF DOMAIN

- Let $D = \mathbb{H} \setminus K$ be a simply connected domain with K compact away from the origin.

- Let γ be an SLE_κ path from 0 to ∞ in D.

- The natural length can be computed two different ways:

  - Considering γ as a curve in $\mathbb{H}$

  - Using the conformal covariance rule.

- The theorem states that these are the same.

OPEN QUESTIONS ON NATURAL LENGTH

- Show that discrete models parametrized by length converge to SLE with natural param.

- Conjecture: the natural length is given (up to constant) by the d-dimensional Minkowski content

$$\Theta_t = c_0 \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \, \text{Area} \, \{z : \text{dist}(z, \gamma_t) < \epsilon\} \, .$$

- Rezaei (in preparation) has shown that the Hausdorff d-dimensional measure is zero. Can we give Hausdorff measure with a guage function?

- As a preliminary step, can we show reversibility of natural length?

SLE_κ IN MULTIPLY CONNECTED DOMAINS FOR $\kappa \leq 4$.

- Conformal invariance and the domain Markov property are not sufficient to characterize $\mu_D^{\#}(z,w)$ for multiply connected domains. This is because the slit domain $D \setminus \gamma_t$ is not conformally equivalent to D.

- (Zhan) For simply connected domains $\mu_D^{\#}(w,z)$ is the same as the reversal of $\mu_D^{\#}(z,w)$. (This has recently been extended to $4 < \kappa < 8$ by Miller-Shefffield. It is not true for $\kappa > 8$.)

- For $\kappa \leq 4$, there is a unique way to define $\mu_D(z,w)$ for multiply connected domains so that conformal covariance and the generalized restriction (boundary perturbation) rule hold.

- The definition (and Zhan's result) immediately show that the measures $\mu(z,w)$ are reversible.

- The definition allows z,w to be interior or boundary points. If $z,w \in \partial D$, they can be in the same component of ∂D (chordal case) or different components (crossing case).

- The definition does not immediately imply that the total mass $\Psi_D(z,w)$ is finite. This has been proved in the cases of doubly connected domains (annuli) and for $\kappa \leq 8/3$ for all domains ($\mathbf{c} \leq 0$). Conjectured to be true for $\kappa \leq 4$. and all domains.

(CROSSING) ANNULUS SLE_κ ($\kappa \leq 4$).

$$A_r = \{r < |z| < 1\}, \quad w = e^{-r+i\theta}.$$

- This is a measure $\mu_{A_r}(1,w)$ on simple paths connecting 1 and w in A_r.

- There exists c_0 such that the partition function satisfies

$$\Psi_{A_r}(1,w) \sim c_0 \, e^{-r(\tilde{b}-b)} \, r^{\mathbf{c}/2}, \quad r \to \infty.$$

- We can get radial SLE_κ (up to multiplicative constant) as a limit

$$\mu_{\mathbb{D}}(1,0) = \lim_{r\to\infty} e^{r(\tilde{b}-b)} \, r^{-\mathbf{c}/2} \, \mu_{A_r}(1, e^{-r}).$$

RADIAL SLE_κ FROM THE INTERIOR POINT

- The probability measure $\mu^{\#}(0,1)$ satisfies the following domain Markov property: given an initial segment γ_t, the distribution on the remainder is annulus SLE in the domain.

- This was derived in a somewhat different way by Zhan.

- Reversibility: Given an initial and a terminal segment, the remainder is annulus SLE_κ in the middle.

COMPARING RADIAL AND WHOLE PLANE SLE_κ ($\kappa \leq 4$) (L.-Field)

- Whole plane SLE_κ is the probability measure on paths $\mu = \mu_{\mathbb{C}}(0, \infty)$ from 0 to ∞ in $\mathbb{C}$ such that given any initial segment γ_t the remainder grows like radial SLE_κ from $\gamma(t)$ to ∞ in the slit domain.

- Suppose that D is a domain containing the origin with a smooth boundary ∂D and $w \in \partial D$.

- Let τ be a stopping time such that $\gamma_\tau \subset D$.

- Expect that $\mu_1 := \mu_D(0, w)$, considered as a measure on γ_τ, is mutually absolutely continuous with respect to μ. Goal: find $d\mu_1/d\mu$.

- Using the generalized restriction rule we would like to write $[d\mu_1/d\mu](\gamma_\tau)$ as

$$\frac{\Psi_{D\setminus\gamma_\tau}(\gamma(\tau),w)}{\Psi_{\mathbb{C}\setminus\gamma_\tau}(\gamma(\tau),\infty)}\ \exp\left\{\frac{\mathbf{c}}{2}\Lambda(\gamma_\tau,\partial D;\mathbb{C})\right\}.$$

- The ratio of partition functions can be calculated by taking $g:\mathbb{C}\setminus\gamma_\tau\to\mathbb{C}\setminus\mathbb{D}$.

$$\frac{|g'(w)|^b\,\Psi_{g(D\setminus\gamma_\tau)}(g(\gamma(\tau)),g(w))}{g'(\infty)^{\tilde{b}}}.$$

  The partition function in the numerator is an annulus partition function.

- However, $\Lambda(\gamma_\tau,\partial D;\mathbb{C})=\infty$.

- Define

$$\Lambda^*(V_1, V_2) = \lim_{r \downarrow 0} \Lambda(V_1, V_2; O_r) - \log\log(1/r),$$

where $O_r = \{|z| > r\}$. Then this limit exists.

- There exists c' such that $[d\mu_1/d\mu](\gamma_\tau)$ equals

$$c' \, \frac{\Psi_{D \setminus \gamma_\tau}(\gamma(\tau), w)}{\Psi_{\mathbb{C} \setminus \gamma_\tau}(\gamma(\tau), \infty)} \exp\left\{ \frac{\mathbf{c}}{2} \Lambda^*(\gamma_\tau, \partial D) \right\}.$$

- We can think of the exponential as a Wick product of a Brownian loop term.

SUMMARY

We can consider SLE_κ as having two ingredients:

- Whole plane SLE_κ
- Brownian loop measure

All other versions can be obtained either as limits, conditional distributions, or by weighting by a Brownian loop measure.