Discretely holomorphic parafermions and integrable boundary conditions

Yacine Ikhlef
Section Mathématiques, Genève

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Outline

1. Introduction
2. Integrable models and isoradial lattices
3. Discrete holomorphicity and integrability

1. Introduction

Discretely holomorphic observables in statistical models

What are they?

- Discretely holomorphic function $F :=$ a function which satisfies a discrete version of Cauchy-Riemann equations $\bar{\partial}F = 0$.
- In stat. mech. models, some correlation functions (*observables*) are discretely holomorphic at criticality.
- Exact analogs of CFT correlators for holomorphic currents.
- In general, F is defined with respect to a lattice interface γ.

Discretely holomorphic observables in statistical models

Why study them?

- Obtain rigorous proofs of CFT results (essentially for Ising)
  - $\gamma \longrightarrow \mathrm{SLE}_\kappa$
  - Scaling limit of correlation functions

  [Chelkak, Duminil-Copin, Hongler, Izyurov, Kytölä, Smirnov]

- Determine the critical Boltzmann weights
  - Proofs for critical values
  - Enhanced numerical studies

  [Duminil-Copin, Smirnov]

  [Beaton, Jensen, de Gier, Guttmann, Lee, Price]

- Relation to integrability

  $\mathrm{O}(n)$, Potts, $\mathbb{Z}_N$, loop models

  [Cardy, Rajabpour, Riva, YI]

Example: the square-lattice Potts model

- Interaction energy:

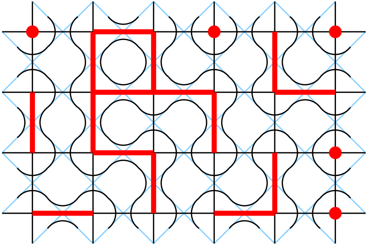
$$E[\sigma] = -J \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j)\,, \qquad \sigma_i \in \{1, \ldots, Q\}\,.$$

- Fortuin-Kasteleyn (FK) cluster representation

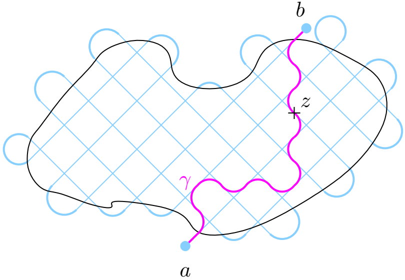
$$W(C) = Q^{\#\text{connected comp.}(C)} \, (e^J - 1)^{\#\text{edges}(C)}$$

"number of states": $Q \in \,]0,4]\,,$ $\begin{cases} \sqrt{Q} := 2\cos\lambda \\ \lambda \in [0, \pi/2[\end{cases}$

- Loops on the medial lattice



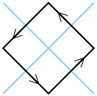
- Boundary conditions on domain Ω



- Lattice observable: $$F_S(z) := \frac{1}{Z} \sum_{C|z\in\gamma} W(C)\, e^{-is\theta_{a\to z}(\gamma)}$$

- (Half-)holomorphicity condition:

$$\sum_{\diamond} F_s(z)\delta z = 0$$



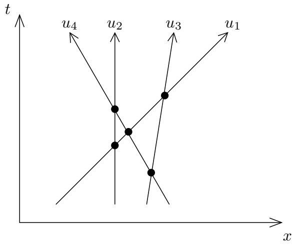
- Determines the spin and the coupling constant:

$$s = 1 - 2\lambda/\pi\,, \qquad J = \log(1+\sqrt{Q})$$

2. Integrable models and isoradial lattices

Factorised scattering

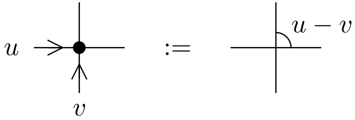
- Scattering of N interacting particles



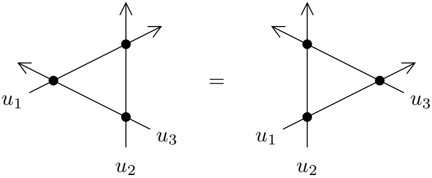
- Space of internal states: $V \Rightarrow$ wavefunction $|\psi(t)\rangle \in V^{\otimes N}$
- Scattering operator: $|\psi_{\mathrm{out}}\rangle = S(u_1, \dots, u_N)\,|\psi_{\mathrm{in}}\rangle$
- Partition function $Z_{\alpha,\beta} := \langle\alpha|\, S(u_1, \dots, u_N)\,|\beta\rangle$

The R-matrix

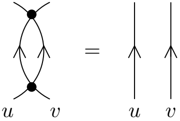
- Local interaction:

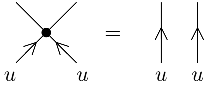
$R(u-v) :=$ 

- Yang-Baxter equation (YBE)



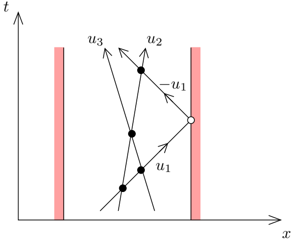
- Inversion relation, regularity



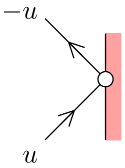


Factorised scattering in the presence of boundaries

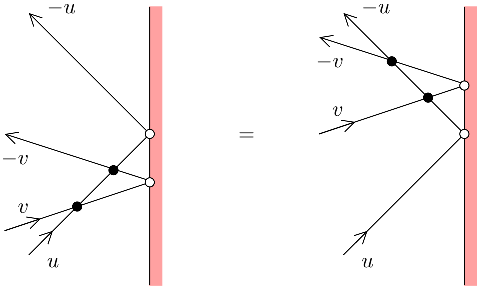
- Scattering of N interacting particles



$K(u) :=$

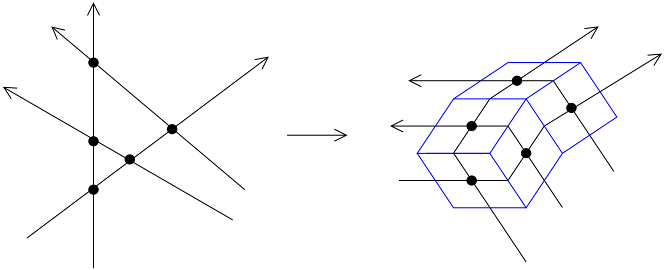


- Boundary Yang-Baxter equation



Isoradial lattices

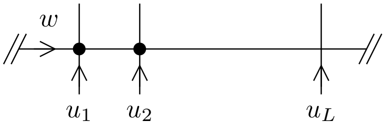
- Particle lines $\rightarrow$ interaction graph (*Baxter lattice*)
- Theorem [Kenyon,Schlenker]:
  The dual of any interaction graph admits an isoradial embedding



- YBE $\equiv$ invariance of Z under translation of any particle line

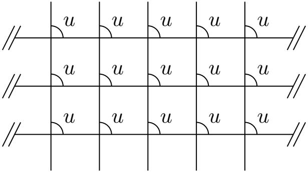
Transfer matrices for the cylinder geometry

- Transfer matrix

$\tau(w) :=$ 

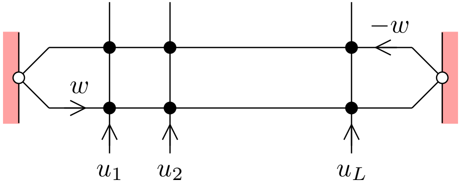
$$\text{YBE} \quad \Rightarrow \quad \tau(w)\tau(w') = \tau(w')\tau(w)$$

- Homogeneous system: $w = u\,, \qquad u_1 = u_2 = \cdots = u_L = 0$



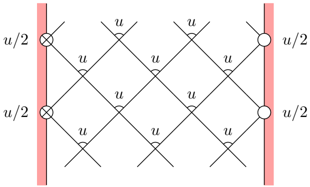
Transfer matrices for the strip geometry

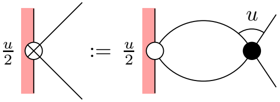
- Transfer matrix

$\tau(w) :=$ 

$$\mathrm{YBE} + \mathrm{BYBE} \quad \Rightarrow \quad \tau(w)\tau(w') = \tau(w')\tau(w)$$

- Homogeneous system:
  $w = u/2\,, \qquad u_1, \ldots u_L = u/2, -u/2, \ldots, u/2, -u/2$



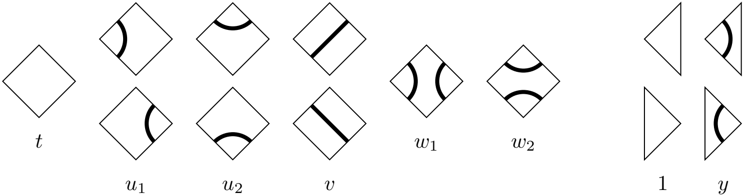


$:= \widetilde{K}(\frac{u}{2})$

Application: critical boundary weights for the $\mathrm{O}(n)$ model

[Yung, Batchelor '94]

- The square-lattice $\mathrm{O}(n)$ model



- Boundary critical behaviour
  - $y < y_c$: ordinary transition
  - $y = y_c$: special transition
  - $y > y_c$: surface transition

- Exact value of y_c?

- Use solution of YBE in the bulk [Nienhuis,Blöte,Batchelor,Warnaar]:

$$\{t(u), u_1(u), \ldots, w_2(u)\} \rightarrow R(u)$$

same universality classes (dense/dilute) as honeycomb $\mathrm{O}(n)$ model (admits 2 additional critical points with an additional Ising variable...)

- Solve BYBE (RKRK=KRKR) and impose $\widetilde{K}(u) = K(u)$

$$y(u) \rightarrow K(u)$$

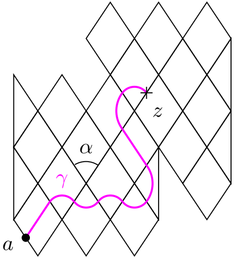
- Find two branches of solutions:
  $K_1 \leftrightarrow$ ordinary transition
  $K_2 \leftrightarrow$ special transition

- Bethe Ansatz $\Rightarrow$ compute surface free energy

3. Discrete holomorphicity and integrability

The O(n) model

- Regular rhombic lattice



- Lattice observable $$F_s(z) := \frac{1}{Z} \sum_{C|\gamma:a\to z} W(C)\, e^{-is\theta(\gamma)}$$
- Bulk discrete Cauchy-Riemann equations $$\sum_{\diamond} F_s(z)\delta z = 0$$
- Solution for the Boltzmann weights [Cardy,YI]

$$t, \ldots, w_2 = t(u = 3\lambda\alpha), \ldots, w_2(u = 3\lambda\alpha) \qquad (n = -2\cos 4\pi\lambda)$$

Boundary constraint

- "Cauchy Riemann" equation on the boundary (suggested by work of [Beaton, de Gier, Guttmann])

$$\mathrm{Re} \sum_{\triangleleft} F_s(z)\delta z = 0$$

- Solution for boundary Boltzmann weight:

$$y = y(u = 3\lambda\alpha/2)$$

- Honeycomb value: $\alpha = \pi/3$
recover results of [Beaton, de Gier, Guttmann]

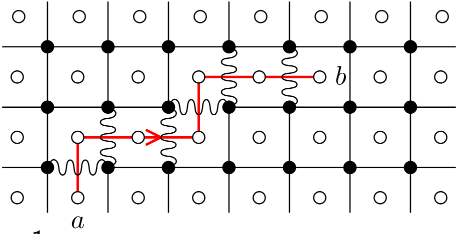
The $\mathbb{Z}_N$ clock model

- The model: $\begin{cases} \sigma_i \in \{1, \omega, \ldots, \omega^{N-1}\} \\ \omega := e^{2i\pi/N} \end{cases}$

  $\mathbb{Z}/N\mathbb{Z}$-invariant interactions

$$\langle \ldots \rangle := \frac{1}{Z} \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} W(\sigma_i^* \sigma_j) \times (\ldots), \qquad W(\sigma^*) = W(\sigma)$$

- Disorder operators

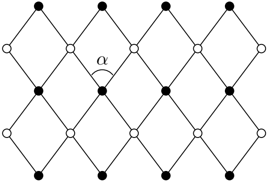


$$\langle \mu_a^* \mu_b \ldots \rangle := \frac{1}{Z} \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle \notin \gamma_{ab}^{\perp}} W(\sigma_i^* \sigma_j) \prod_{\langle ij \rangle \in \gamma_{ab}^{\perp}} W(\omega \sigma_i^* \sigma_j) \times (\ldots)$$

Prop: $\langle \mu_a^* \mu_b \rangle$ independent of γ_{ab}

Discrete parafermions [Cardy,Rajabpour]

- (Deformed) covering lattice



- Parafermion operator: $\psi(z) := \sigma_{z'} \times \mu_{z''} \times e^{-is\theta(z',z'')}$
- Observable (for fixed $a \in \partial\Omega$):

$$F_s(z) := \langle \psi^*(a)\psi(z) \rangle$$

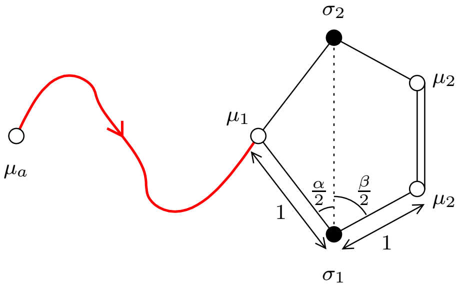
- Solution of discrete CR equation:

$$\sum_{\diamond} F_s(z)\delta z = 0 \quad \Leftrightarrow \quad \begin{cases} s = 1 - 1/N \\ W(\sigma) = W\left(u = \frac{\pi-\alpha}{2N} | \sigma\right) \end{cases}$$

- $W(u|\sigma) :=$ self-dual integrable weights of [Fateev,Zamolodchikov]

Boundary constraint

- Boundary weights: $Y(\sigma_i^* \sigma_j)$ (preserve $\mathbb{Z}_N$ invariance)
- Boundary plaquette:



- Boundary Cauchy-Riemann equation

$$\sum_{\diamond} \left[e^{i\varphi} F_s(z)\delta z + e^{-i\varphi} \widetilde{F}_s^*(z)\delta z^* \right] = 0$$

where $\widetilde{F}_s := F_s[\sigma \to \sigma^*]$

- Consistency conditions on Y $\Rightarrow$ $\varphi = \pi/N$

Solution of boundary CR equation

- Use discrete Fourier transform

$$\widehat{Y}(u,\xi|k) = W(u+\xi|k)W(u-\xi|k)$$

- Relation to angles

$$u = \frac{\pi-\alpha}{4N}, \qquad \xi = \frac{\pi-\beta}{4N}$$

- Satisfy boundary YBE!
- $\Rightarrow$ found new non-trivial solution of BYBE in $\mathbb{Z}_N$
- For $\xi = u$: free BC

Conclusions

- Summary
  - "Boundary CR equation" := a *local*, *linear* condition for lattice critical BCs
  - [Solution of BCR] $\equiv$ [Solution of BYBE with $u \to u/2$]
  - In $\mathrm{O}(n)$ and $\mathbb{Z}_N$ models: found critical BCs invariant under internal symmetry
- Open questions
  - Relate BCR to some condition in boundary CFT?
  - Classification of conf. inv. BCs in $\mathbb{Z}_N$-parafermion CFT?
  - Extend to non-invariant critical BCs (ex: "blobbed" $\mathrm{O}(n)$)?
  - Useful for mathematical proofs?

Thank you for your attention!