A generalised identity for SAW on the honeycomb lattice

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Hungarian-Finnish

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- Ice under in-winter living fish swim
- Jég alatt télen eleven halak uszkálnak.
- Jään alla talvella elävät kalat uiskentelevat.
- voj (H), voi (F) (butter),
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References

- The critical fugacity for surface adsorption of SAW on the honeycomb lattice is $1 + \sqrt{2}$. arXiv 1109.0358 Commun. Math. Phys, (shortly to be replaced with a version with a correct proof!).
- A numerical adaptation of SAW identities from the honeycomb to other two-dimensional lattices. arXiv 1110.1141, J. Phys. A
- Two-dimensional SAW and polymer adsorption: Critical fugacity estimates. arXiv 1110.6695, J Phys A
- Off-critical partition functions and the winding angle distribution of the O(n) model. arXiv 1203.2959, J Phys A

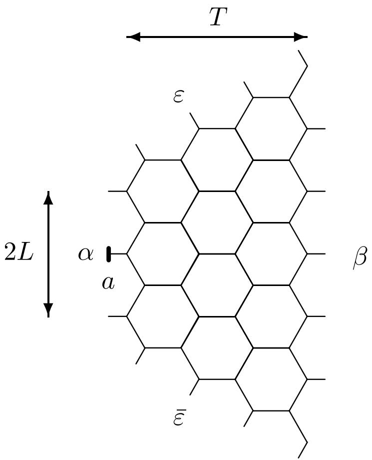


Figure: The figure shows the domain of width T and height $2L$. Walks start at point a and finish internally, or on the α, β or ε ($\bar{\varepsilon}$) wall.

Let

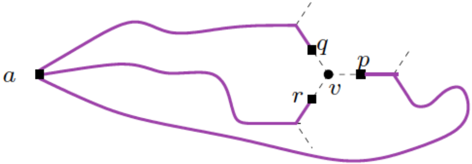
$$F(p) \equiv F(t,\alpha;p) = \sum_{\omega: a \rightsquigarrow p \text{ in } D} t^{|\omega|} e^{i\alpha W(\omega)},$$

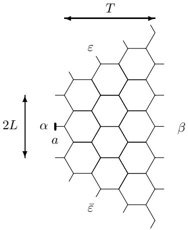
If p, q and r are the 3 mid-edges around a vertex v of the honeycomb lattice, taken in counterclockwise order, then, for $t = t_c$ and $\alpha = -5/8$,

$$F(p) + jF(q) + j^2 F(r) = 0,$$

where $j = e^{2i\pi/3}$, or, more symmetrically,

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0.$$

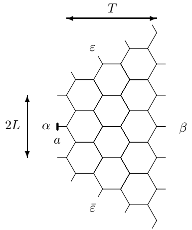




Sum $(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0$, a local identity, over all vertices in the domain.

- The inner mid-edges don't contribute.
- The domain has a N-S symmetry
- The winding number of walks hitting the boundary is known

$$\cos(\frac{3\pi}{8})A_{T,L}(x_c) + B_{T,L}(x_c) + \cos(\frac{\pi}{4})E_{T,L}(x_c) = 1.$$



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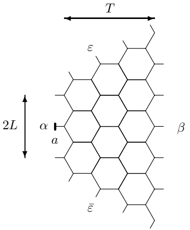

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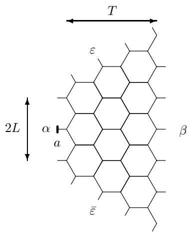

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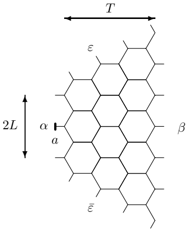

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$$\cos(\frac{3\pi}{8})A_{T,L}(x_c) + B_{T,L}(x_c) + \cos(\frac{\pi}{4})E_{T,L}(x_c) = 1.$$

- By first letting $L \to \infty$, then $T \to \infty$, D-C/S showed that the ogf of half-plane SAW diverges at x_c, establishing that $\mu \geq \sqrt{2+\sqrt{2}}$.
- Then by considering bridges they showed that the ogf for half-plane SAW is finite for $x < x_c$, and so $\mu \leq \sqrt{2+\sqrt{2}}$.
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Remarkable nature of the result

- Letting $L \to \infty$, it follows that $E_T(x_c) = 0$.
- Then in a strip of width T we have

$$\cos(\frac{3\pi}{8})A_T(x_c) + B_T(x_c) = 1.$$

- Consider a strip of width 0, i.e. a spine. Then

$$A_T(x) = \frac{2x^3}{1-x^2}, \quad B_T(x) = \frac{2x^2}{1-x^2}.$$

- Then solve

$$\cos(\frac{3\pi}{8})A_T(x) + B_T(x) = 1.$$

- The solution is $x_c = 1/\sqrt{2+\sqrt{2}}$.

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Generalise to O(n) model

Consider the O(n) honeycomb loop model, with $n \in [-2, 2]$.

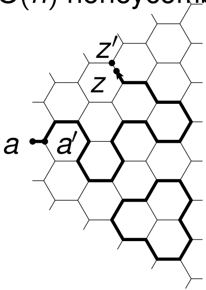


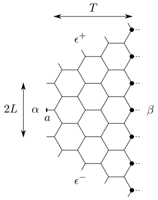
Figure: a and z are mid-edges and a' and z' are their vertices. The contribution to $F(z)$ is $\mathrm{e}^{-2\mathrm{i}\sigma\pi/3}x^{30}n$.

With $a \in \partial\Omega$, $z \in \Omega$, and γ including loops, set

$$F_H(\Omega, a, z; x, n, \sigma) := F_H(z) = \sum_{\gamma(a\to z)\subset\Omega} \mathrm{e}^{-\mathrm{i}\sigma W(\gamma(a\to z))} x^{\ell(\gamma)} n^{c(\gamma)},$$

Surface interactions

Take the same domain $S_{T,L}$, but add weights to the vertices on the β boundary:



Then we find ($y* = 1 + \sqrt{2}$ (Batchelor/Yung '95 $y_c = y*$)

$$1 = \cos\left(\frac{3\pi}{8}\right) A_{T,L}(x_c, y) + \cos\left(\frac{\pi}{4}\right) E_{T,L}(x_c, y) + \frac{y^* - y}{y(y^* - 1)} B_{T,L}(x_c, y)$$

where y index counts visits to the β boundary.

Safe to take $L \to \infty$: (so we have a strip)

$$1 = \cos\left(\frac{3\pi}{8}\right) A_T(x_c, y) + \cos\left(\frac{\pi}{4}\right) E_T(x_c, y) + \frac{y^* - y}{y(y^* - 1)} B_T(x_c, y)$$

Straightforward to show

- $E_T(x_c, y) = 0$ for $0 \le y < y^*$
- $y_c \ge y^*$
- $\lim_T A_T(x_c, y) = A(x_c, y) = A(x_c)$ is constant for $0 \le y < y^*$

So write

$$\cos(3\pi/8) A(x_c, y) = 1 - \delta$$

Then the identity reduces to

$$B(x_c, y) = \lim_T B_T(x_c, y) = \frac{\delta y(y^* - 1)}{y^* - y}$$

and in particular

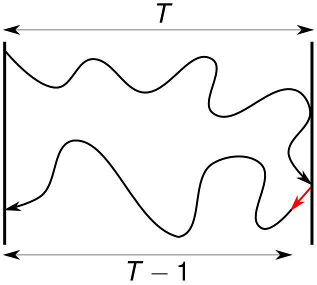
$$B(x_c, 1) = \delta$$

Proof of the critical fugacity

Proposition

If $\delta = 0$ then $y_{\rm c} = y^*$.

Proof uses a decomposition of A_T walks into B_T walks:



$$A_T(x_{\rm c}, y) - A_{T-1}(x_{\rm c}, 1) \leq x_{\rm c} B_{T-1}(x_{\rm c}, 1) B_T(x_{\rm c}, y),$$

The proof of $\delta = 0$ is complicated! (very probabilistic!)

- Use renewal theory to show δ^{-1} is the expected height of an irreducible bridge
- Show $\mathbb{E}[\mathit{height}] < \infty \Rightarrow \mathbb{E}[\mathit{width}] < \infty$
- Show $\mathbb{E}[\mathit{width}] < \infty$ leads to a contradiction

Due to Hugo Duminil-Copin.

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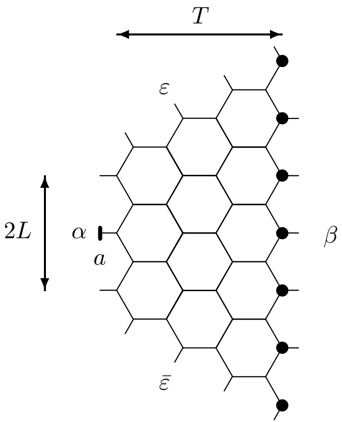

Figure: Finite patch $S_{3,1}$ of the honeycomb lattice with a boundary. The SAW starts at a, ends at a mid-edge z, acquiring a weights x for each step, and y for each contact (filled circle) with the right hand side boundary.

- There is no corresponding equation for SAW on other lattices.
- For the square lattice, Cardy and Ikhlef found a similar (parafermionic) observable, but the model describes osculating SAW with asymmetric weights.
- Arguing that the scaling limit of all two-dimensional SAW models should be identical, “something similar" should be true for SAW on other lattices.
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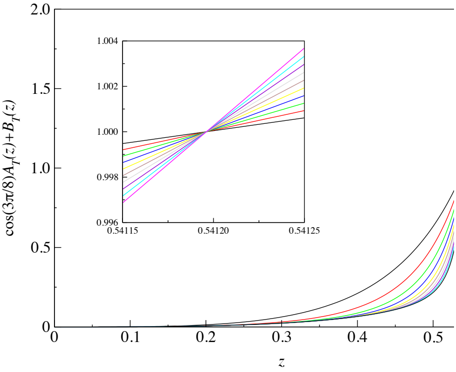

Figure: Bad picture with nice inset of $c_\alpha A_T(x) + B(x)$ for honeycomb lattice walks in a strip of width $1, \cdots, 10$.

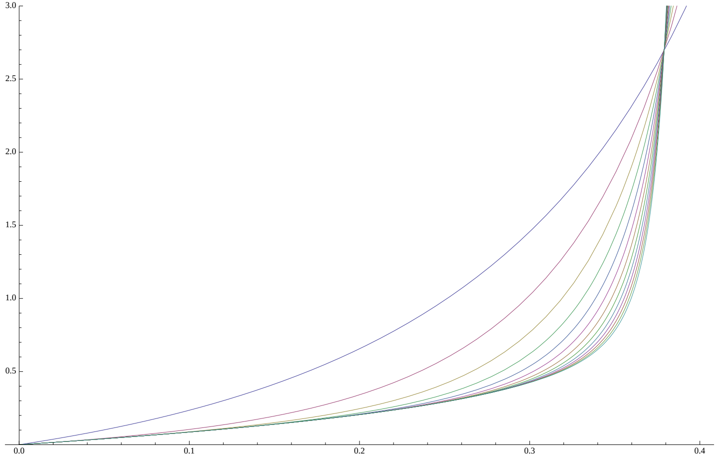

Figure: Square lattice $c_\alpha A_T(x) + B(x)$ for walks in a strip of width $1, \cdots, 15$.

Conjecture (best estimates of x_c):

$$1 = c_A(T)A_T(x_c) + c_B(T)B_T(x_c),$$

Successive widths $(T, T+1)$ give $c_A(T)$ and $c_B(T)$.
(Square lattice $T \leq 15$, triangular lattice $T \leq 11$).
Extrapolate:

$$\lim_{T\to\infty} \frac{c_A(T)}{c_B(T)} = \cos\left(\frac{3\pi}{8}\right)$$

to 6 sig. digits. Hence

$$\cos\left(\frac{3\pi}{8}\right) A_T(x_c) + B_T(x_c) = const. + correction$$

In fact $1.02497(1 - 0.14/T^2)$, similarly for the triang. lattice.

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More generally, assume

$$c_A(T)A_T(x_c)+c_B(T)B_T(x_c) = c_A(T+1)A_{T+1}(x_c)+c_B(T+1)B_{T+1}(x_c)$$

Successive triples give

$$c_A(T),\ c_B(T),\ x_c(T).$$

Extrapolate $x_c(T)$ and find
$x_c(sq) = 0.37905228(1)$ and $x_c(tr) = 0.240917575(10)$.
(Since used for honeycomb NASAW).

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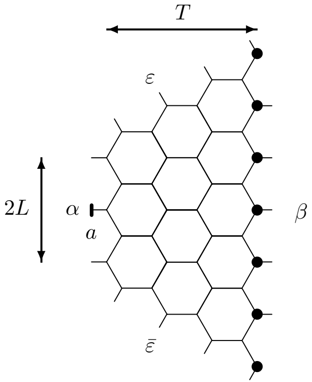

Figure: Finite patch with a boundary. The SAW acquires weights x, y for each step/contact.

$$\cos(\frac{3\pi}{8})A_T(x_c, y) + \cos(\frac{\pi}{4})\left(\frac{y_c - y}{y}\right) B_T(x_c, y) = 1.$$

- At $y = y_c$ we have

$$\cos(\frac{3\pi}{8})A_T(x_c, y) = 1.$$

- So for the honeycomb lattice $T = 1$ and $T = 2$ results are enough to calculate x_c and y_c!
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- So for the honeycomb lattice $T = 1$ and $T = 2$ results are enough to calculate x_c and y_c!
- Other lattices?
- Vertex or site interaction?
- Honeycomb all vertices interaction ?

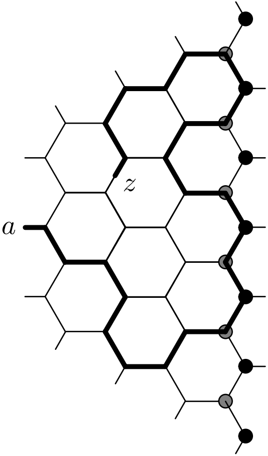

Figure: The figure shows the two types of surface sites on the honeycomb lattice as indicated by solid and shaded circles.

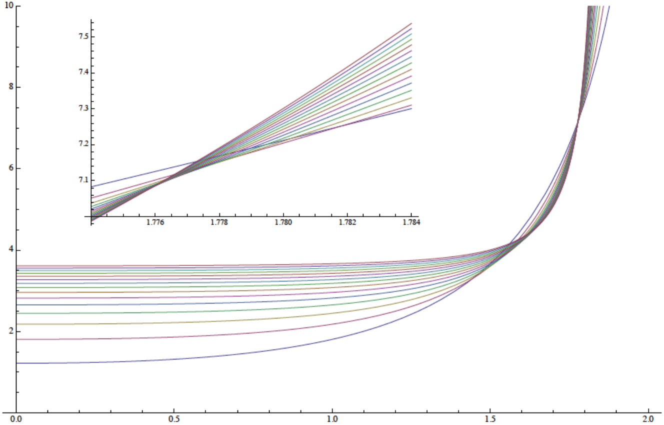

Figure: Square lattice with surface site interactions. $A_T(x_c, y)$ versus y for $T = 1 \ldots 15$. Inset shows the intersection region in finer scale.

- We denote by $y_c(T)$ the point of intersection of $A_T(x_{\rm c}, y)$ and $A_{T+1}(x_{\rm c}, y)$.
- We observe that the sequence $\{y_c(T)\}$ is a monotone function of T. The argument above implies that $\lim_{T\to\infty} y_c(T) = y_c$.
- This then suggests a new numerical approach to estimating y_c.
- One first calculates the generating functions $A_T(x_{\rm c}, y)$, for all strip widths $T = 0, 1, 2, \ldots T_{\rm max}$.
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Table: Estimated critical fugacity y_c for surface adsorption.

Lattice	Site weighting	Edge weighting
Honeycomb	1.46767	$\sqrt{1+\sqrt{2}}$
Square	1.77564	2.040135
Triangular	2.144181	2.950026

For the square lattice these are at least 1000 times more accurate than other methods. Other results are new.

Extend off-criticality

- We redefine the parafermionic operator ($\bar{\sigma} = 1 + \sigma$) as

$$F(x) = F(a, x, z, \bar{\sigma}) = \sum_{\gamma \subset \Omega : a \to x} e^{\bar{\sigma} W(\gamma)} z^{|\gamma|}.$$

- In terms of this redefined operator, the D-C/S result is

$$\sum_{\gamma \subset \Omega : a \to x} e^{i \frac{3}{8} W(\gamma)} z_c^{|\gamma|} = 1.$$

where the sum is over all walks starting at a and ending at x, on the boundary of Ω.

Theorem

For $z \leq z_c$

$$\sum_{\gamma:a\to x\in\partial\Omega} e^{\frac{3i}{8}W(\gamma)} z^{|\gamma|} + (1 - z/z_c) \sum_{\gamma:a\to x\in\Omega\backslash\partial\Omega} e^{\frac{3i}{8}W(\gamma)} z^{|\gamma|} = 1.$$

The first sum is over all walks that finish at the surface of the domain, while the second sum is over all walks that finish strictly in the interior of the domain.

We do this for the n-vector model $n \in [-2, 2]$. The r.h.s. becomes a loop generating function, and F has an extra parameter that counts loops.

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Sketch of proof

Lemma For a given vertex v with mid-edges p, q and r, and z below the critical value z_c, the observable $F_H(z)$ satisfies

$$(p-v)F_H(p)+(q-v)F_H(q)+(r-v)F_H(r)=(1-\frac{z}{z_c})F_V(v)$$

where

$$F_V(v):=(p-v)F_H(p;v)+(q-v)F_H(q;v)+(r-v)F_H(r;v)$$

and $F_H(p;v)$ only includes configurations where there is a walk terminating at the mid-edge p which precedes the vertex v.
Now sum over all vertices, using this lemma, and the result follows, after invoking symmetries, and some simplifications.

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where

$$F_V(v) := (p - v)F_H(p; v) + (q - v)F_H(q; v) + (r - v)F_H(r; v)$$

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Now sum over all vertices, using this lemma, and the result follows, after invoking symmetries, and some simplifications.

Exponent inequalities

In a strip, we obtain

$$\cos(\frac{3\pi}{8})A_T(z) + B_T(z) + (1 - z/z_c)G_T(z) = 1.$$

As $T \to \infty$, $A_T(z) \to \chi_{11}(z)$, $B_T(z) \to 0$, and $G_T(z) \le \chi_1$.

It immediately follows that

$$\gamma_{11} \le \gamma_1 - 1.$$

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Winding angle calculation

Now to obtain the winding angle distribution function directly from the off-critical generating function identity.
Define

$$G_{\theta,\Omega}(x) = \sum_{\substack{\gamma: a \to z \in \Omega \setminus \partial\Omega \\ W(\gamma)=\theta}} x^{|\gamma|} n^{c(\gamma)}$$

(contributions to $G_\Omega(x)$ that involve walks with winding angle θ.
Also

$$F_\Omega(x) = \sum_{\gamma: a \to z \in \partial\Omega} e^{\tilde{\sigma} i W(\gamma)} x^{|\gamma|} n^{c(\gamma)},$$

the ogf of “walks" that end on a domain boundary.
(To simplify the notation, γ now describes a walk and a configuration of loops.)

The key identity becomes

$$F_\Omega(x) + (1 - x/x_c)\sum_\theta e^{\tilde{\sigma} i\theta} G_{\theta,\Omega}(x) = C_\Omega(x)$$

Normalise, so that $F^*_\Omega(x) = \dfrac{F_\Omega(x)}{C_\Omega(x)}$ and $G^*_{\theta,\Omega}(x) = \dfrac{G_{\theta,\Omega}(x)}{C_\Omega(x)}$.

For $x < x_c$ define $F^*(x)$ and $G^*_\theta(x)$ to be $F^*_\Omega(x)$ and $G^*_{\theta,\Omega}(x)$ respectively as Ω approaches the half plane. So

$$F^*(x) + (1 - x/x_c)\sum_\theta e^{\tilde{\sigma} i\theta} G^*_\theta(x) = 1. \qquad (1)$$

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$$F^*(x) \sim 1 + const.(1 - x/x_c)^{-\gamma_{11}}; \;\; \sum_\theta G^*_\theta(x) \sim const.(1 - x/x_c)^{-\gamma_1}.$$

From equation (1) we obtain

$$\sum_\theta e^{\tilde{\sigma} i\theta} G^*_\theta(x) = \frac{1 - F^*(x)}{1 - x/x_c} \sim C(1 - x/x_c)^{-\gamma_{11}-1}. \qquad (2)$$

$$\text{Let } \; G^*_\theta(x) = \sum_{j=0}^{\infty} a_\theta(j) x^j.$$

Then

$$\sum_\theta a_\theta(j) \sim const. x_c^{-j} j^{\gamma_1 - 1}, \;\text{and}\; \sum_\theta e^{\tilde{\sigma} i\theta} a_\theta(j) \sim C x_c^{-j} j^{\gamma_{11}}.$$

$$\sum_\theta a_\theta(j) \sim const.x_c^{-j} j^{\gamma_1 - 1}, \text{ and } \sum_\theta e^{\tilde{\sigma} i\theta} a_\theta(j) \sim C x_c^{-j} j^{\gamma_{11}}.$$

$$\text{Let } b_\theta(j) = \frac{a_\theta(j)}{\sum_\theta a_\theta(j)}.$$

$$\sum_\theta e^{\tilde{\sigma} i\theta} b_\theta(j) \sim const.j^{\gamma_{11} - \gamma_1 + 1}$$

Now b_θ is just the probability distribution function $P(\theta)$, so

$$\sum_\theta e^{\tilde{\sigma} i\theta} b_\theta(j) \approx \int_{-\infty}^{\infty} e^{\tilde{\sigma} i\theta} P(\theta) d\theta \propto j^{-\omega}.$$

So

$$1 + \gamma_{11} - \gamma_1 = -\omega.$$

This gives the winding angle distribution exponent for all O(n) models with $n \in [-2, 2]$ in terms of exponents γ_1 and γ_{11}.

From the existing physics literature, one has (CFT)

$$\gamma_1 = \frac{\kappa^2 + 12\kappa - 12}{8\kappa(4-\kappa)}, \qquad \gamma_{11} = -\frac{2(3-\kappa)}{\kappa(4-\kappa)},$$

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Winding angle distribution

Obtained by Duplantier and Saleur for the O(n) model. We have $n = -2\cos(4\pi/\kappa)$. They conjecture, from CFT and Coulomb gas arguments, that

$$P(x=\theta) \propto \exp\left(-\frac{\theta^2}{2\kappa\nu\log\ell}\right), \qquad \ell \to \infty,$$

where ℓ is the length of the SAW. ν is the standard critical exponent relating the circumference of the cylinder to the length of the walk, $L \sim \ell^\nu$, $\nu = \frac{1}{4-\kappa}$. It follows that

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- Note that for SAW ($\kappa = 8/3$) this gives $\omega = \tilde{\sigma}^2 = \frac{9}{64}$.
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