Mappings of the 2-dimensional Ising model

Béatrice de Tilière
Université Pierre et Marie Curie, Paris

j.w. with Cédric Boutillier

Helsinki
June 12, 2012

2-dimensional Ising model

- Let $G = (V, E)$ be a finite planar graph.
- Spin configuration: $\sigma \in \{-1, 1\}^V$.
- Each edge e is assigned a coupling constant, $J_e \geq 0$.
- Weight of spin configuration σ:

$$e^{\beta \sum_{\{e=uv \in E\}} J_e \sigma_u \sigma_v},$$

where β is the inverse temperature.
- Ising Boltzmann measure:

$$P_{\text{Ising}}(\sigma) = \frac{1}{Z_{\text{Ising}}} e^{\beta \sum_{\{e=uv \in E\}} J_e \sigma_u \sigma_v},$$

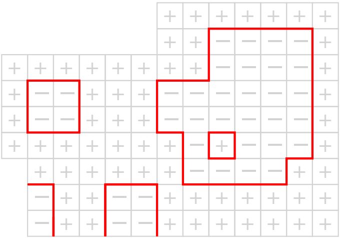
where $Z_{\text{Ising}} = \sum_{\sigma \in \{-1,1\}^V} e^{\beta \sum_{\{e=uv \in E\}} J_e \sigma_u \sigma_v}$ is the partition function.

Low temperature expansion

- Ising model on G^*, coupling constants (J_{e^*}), free boundary conditions.
- Low temperature expansion of the partition function

$$Z_{\text{Ising}} = C \sum_{P \in \mathcal{P}^{\text{even}}(G)} \prod_{e \in P} e^{-2J_{e^*}}.$$

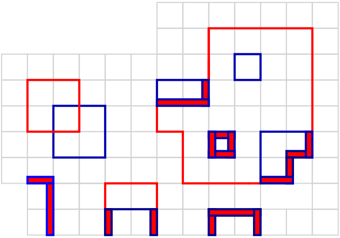
$\Rightarrow$ Polygonal configurations $\mathcal{P}^{\text{even}}(G)$, separating clusters of spins, with an even number of edges incident to the boundary.



Low temperature representation of a configuration σ

Low temperature expansion of 2 Ising models

- Two Ising model on G^* with
  - free boundary conditions
  - same spins along the inside of the boundary
  - independent elsewhere
- Low temperature expansion ⇒ superimposition of polygonal configurations such that:
  - no monochromatic edge is incident to the boundary
  - even number of bi-chromatic edges are incident to the boundary



Low temperature representation of 2 configurations σ, σ'

XOR Ising model [Wilson]

From 2 spin configurations σ, σ' as above, construct the XOR-spin configuration ξ:

$$\forall v^* \in G^*, \quad \xi(v^*) = \sigma(v^*)\sigma'(v^*).$$

Remark

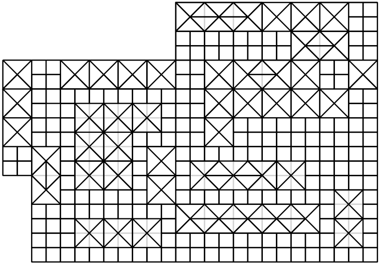
Monochromatic edges correspond to edges of the low temperature expansion of ξ. They are polygonal contours not touching the boundary of G.

Conjecture [Wilson] XOR loops are contour lines of a Gaussian free field, but there are $\sqrt{2}$ fewer loops than for the double dimer model on the square lattice or honeycomb lattice.

Theorem (Boutillier, dT)

Monochromatic loop configurations have the same distribution as:

1. *a (restriction) of loop configurations in the* 6*-vertex model.*
2. *a (restriction) of the level lines of the height function of a bipartite dimer model.*



Known mapping 1 (Baxter, Dubédat, Kramers, Lieb, Wannier, Wu)

2 independent Ising model on G and G^*

$\Downarrow$ low temperature expansion + abelian duality

8-vertex model on the medial-graph

Known mapping 1 (Baxter, Dubédat, Kramers, Lieb, Wannier, Wu)

| 2 independent Ising model on G and G^* | + K-W self-duality

$\Downarrow$ low temperature expansion + abelian duality

| 6-vertex model | on the medial graph, free-fermionic point

$\Downarrow$

| Bipartite dimer model |

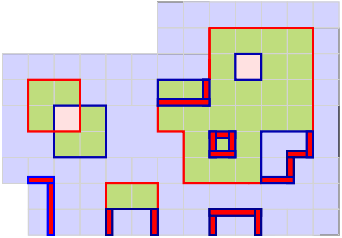
- $\oplus$ Tracking of order and disorder variables $\Rightarrow$ Computation of critical correlators [Dubédat], see also [Chelkak, Hongler, Izyurov].
- $\ominus$ Loosing track of configurations

Mixed contour expansion

Monochromatic polygon configuration P separate G into faces $F_1, \cdots, F_{n(P)}$.

Lemma

Inside each face F_i, bichromatic edges are polygon configurations with an even number of edges incident to the boundary.



Mixed contour expansion

Lemma

Given a polygon configuration P and for each face $F_1, \cdots, F_{n(P)}$, polygon configurations of bichromatic edges as above, there are $2.2^{n(P)}$ corresponding pairs of spin configurations. As a consequence:

$$Z_{\text{Ising}}^2 = C \sum_{P \in \mathcal{P}^0(G)} \Big(\prod_{e \in P} e^{-2J_{e^*}} \Big) 2^{n(P)} \prod_{i=1}^{n(P)} \underbrace{\Big(\sum_{P' \in \mathcal{P}^{\text{even}}(F_i)} \prod_{e \in P'} e^{-4J_{e^*}} \Big)}.$$

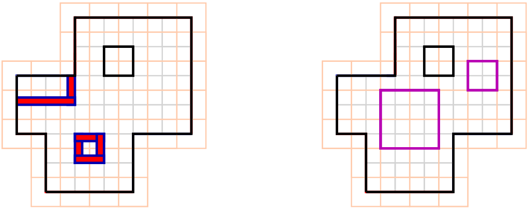
The term $\underbrace{\quad}$ is the low temp. expansion of an Ising model:

- on F_i^*, the smallest dual face containing F_i,
- with free boundary conditions,
- with coupling constants $(2J_{e^*})$.

Mixed contour expansion

Kramers-Wannier high temperature expansion yields:

$$\sum_{P' \in \mathcal{P}^{\text{even}}(F_i)} \prod_{e \in P'} e^{-4J_{e^*}} = C(F_i) \sum_{P \in \mathcal{P}^0(F_i^*)} \prod_{e^* \in P^*} \tanh(2J_{e^*}).$$



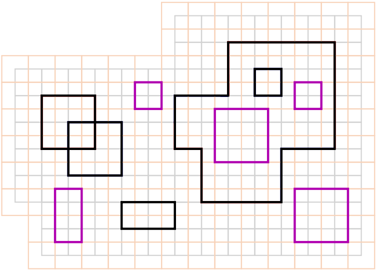
Low temp. expansion on G. High temp. expansion on G^*

Mixed contour expansion

Combining terms, and taking $C(F_i), i \in \{1, \cdots, n(P)\}$ into account:

$$Z^2_{\text{Ising}} = C \sum_{P \in \mathcal{P}^0(G)} \prod_{e \in P} \left(\frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}} \right) \prod_{i=1}^{n(P)} \sum_{P^* \in \mathcal{P}^0(F_i^*)} \prod_{e^* \in P^*} \left(\frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}} \right)$$

$$= C \sum_{\{(P, P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*) :\, P \cap P^* = \emptyset\}} \prod_{e \in P} \left(\frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}} \right) \prod_{e^* \in P^*} \left(\frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}} \right)$$

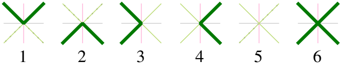


6-vertex model on the medial graph

- Medial graph G^M of G (or G^*):
    - vertices of G^M $\leftrightarrow$ edges of G (or G^*),
    - vertices are joined by an edge if corresponding edges are incident.

Remark: vertices of the medial graph have degree 4.

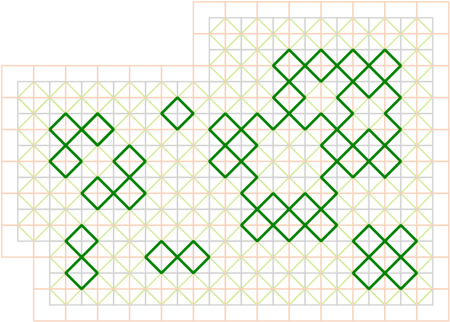
- 6-vertex configuration: at each vertex, one of the following:



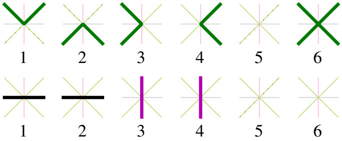
- Weights: $\omega_{12} = \frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}$, $\omega_{34} = \frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}$, $\omega_{56} = 1$.
  Satisfy the free fermion condition : $\omega_{12}^2 + \omega_{34}^2 = \omega_{56}^2$.

6-VERTEX MODEL ON THE MEDIAL GRAPH

6-vertex configuration.

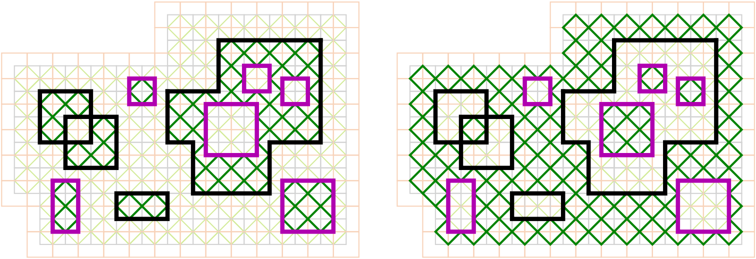


Mapping:



Lemma

- *6-vertex configuration* $\rightarrow$ *a pair* (P, P^*) *of primal and dual polygonal configurations of* G *and* G^**:*
  - P *and* P^* *do not intersect,*
  - P *and* P^* *do not touch the boundary.*
- *Given* (P, P^*) *as above* $\rightarrow$ *two 6-vertex configurations.*



Corollary

As a consequence:

- $Z_{\text{Ising}}^2 = C_1\, Z_{6\text{V}}\,,$
- *Monochromatic polygonal configurations of the double Ising model have the same distribution as primal (black) polygonal configurations in the* 6*-vertex model.*

Dimer model

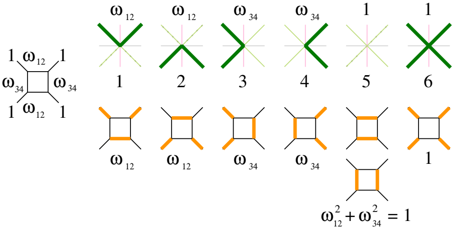
- Planar graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$.
- Dimer configuration: subset of edges M such that each vertex is incident to exactly on edge of M.
- Every edge e is assigned a positive weight, ν_e.
- Dimer Boltzmann measure (finite graph):

$$P_{\text{dimer}}(M) = \frac{1}{Z_{\text{dimer}}} \prod_{e \in M} \nu_e,$$

where Z_{dimer} is the partition function.

From 6-vertex to a bipartite dimer model

Mapping [Dubédat]



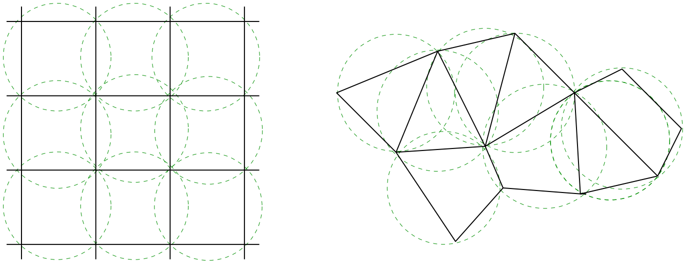
Corollary (Boutillier-dT)

- $Z_{\text{Ising}}^2 = C_2 \, Z_{\text{quadri}}$.
- *Monochromatic polygonal configurations of the double-Ising model have the same distribution as configurations corresponding to primal polygonal configurations in the bipartite dimer model.*

Then, define the height function of bipartite dimer model so that the level lines of its restriction to primal vertices are primal loops of the 6-vertex model.

Isoradial graphs

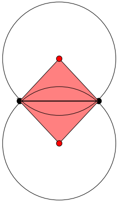
- A graph $G = (V, E)$ is isoradial if it can be embedded in the plane such that all faces are inscribed in a circle of radius 1.

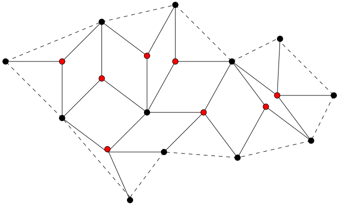


- G isoradial $\Rightarrow G^*$ isoradial: vertices of G^* = center of the circumcircles.

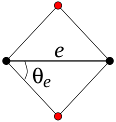
Isoradial graphs

- $G^\diamond$: associated rhombus graph, $\begin{cases} \text{vertices: } V(G) \cup V(G^*) \\ \text{edges: radii of the circles} \end{cases}$





- Edge $e \rightarrow \begin{cases} \text{rhombus (or } 1/2) \\ \text{angle } \theta_e \end{cases}$

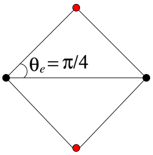


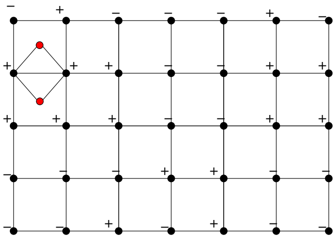
Critical 2-dimensional Ising model (Baxter)

- The Ising model on an isoradial graph is critical if the coupling constants are given by, for every edge e:

$$J_e = \frac{1}{2} \log \left(\frac{1 + \sin \theta_e}{\cos \theta_e} \right).$$

- Example: $G = \mathbb{Z}^2$. For every edge e, $\theta_e = \frac{\pi}{4}$, $J_e = \frac{1}{2} \log(1 + \sqrt{2})$.

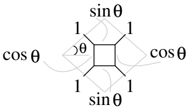




$\Rightarrow$ critical temperature computed by Kramers and Wannier.

Critical quadri-tilings

- When G is isoradial (critical or not), the bipartite dimer model corresponding to the double Ising model is the quadri-tilings model (studied in [dT, thesis]): random interface model in dimensions $2+2$.
- When the Ising model is critical, weights of quadri-tilings are:



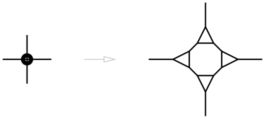
Theorem (dT)

The height function of critical quadri-tilings on an isoradial graph (in the plane) converges weakly in distribution to $1/\sqrt{\pi}$ *a Gaussian free field.*

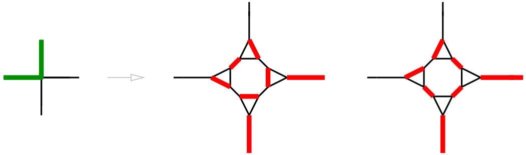
Also, have results when the underlying rhombus lattice is random (random rhombus tiling of the triangular lattice).

Ising-dimer correspondence (Fisher)

- Low temperature of Ising model on a toroidal graph G^*, coupling constants J_{e^*}.
- Fisher graph $\mathcal{G}$ of G.



- Correspondence: polygonal config. $\rightarrow 2^{|V|}$ dimer config.



- Critical Ising model on $G \rightarrow$ critical dimer model on $\mathcal{G}$.

$$\nu_e = \begin{cases} \cot \frac{\theta_e}{2} & \text{shared edges} \\ 1 & \text{new edges.} \end{cases}$$

Kasteleyn matrix

- Kasteleyn matrix
  - Kasteleyn orientation of the edges.
  - K is the weighted, oriented, adjacency matrix of the graph.

$$K_{u,v} = \begin{cases} \nu_{uv} & \text{if } u \sim v,\ u \to v \\ -\nu_{uv} & \text{if } u \sim v,\ u \leftarrow v \\ 0 & \text{otherwise.} \end{cases}$$

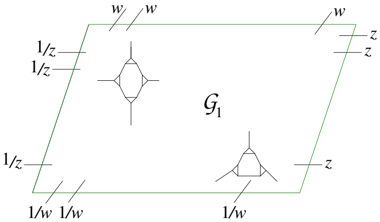
Theorem (Kasteleyn, Temperley and Fisher)

$$Z_{dimer} = |\mathrm{Pf}(K)|.$$

- Remark: $\mathrm{Pf}(K)^2 = \det(K) \Rightarrow \det(K)$ counts double dimer configurations

Critical dimer model on infinite periodic Fisher graph

- $\mathcal{G}$: infinite $\mathbb{Z}^2$-periodic, Fisher graph.
- Toroidal exhaustion $\{\mathcal{G}_n\} = \{\mathcal{G}/n\mathbb{Z}^2\}$. $\mathcal{G}_1$: fundamental domain.
- Key object: characteristic polynomial (free energy, Gibbs measure).
  - K: Kasteleyn matrix of $\mathcal{G}$ with critical weight function.
  - $K_1(z,w)$: Kasteleyn matrix of $\mathcal{G}_1$, with weights modified on edges crossing a horizontal and vertical path in the dual graph.



- Dimer characteristic polynomial: $\det K_1(z,w)$.

Characteristic polynomial and the Laplacian

- G: infinite, isoradial, $\mathbb{Z}^2$-periodic graph.
- Laplacian on G, with weights $\tan\theta_e$ is represented by the matrix Δ:

$$\Delta_{u,v} = \begin{cases} \tan\theta_{uv} & \text{if } u \sim v \\ -\sum_{w\sim u} \tan\theta_{uw} & \text{if } u = v. \end{cases}$$

- $\Delta_1(z,w)$: Laplacian matrix on G_1 with weights z and w.
- Laplacian characteristic polynomial: $\det \Delta_1(z,w)$.

Theorem (Boutillier,dT)

There exists a constant $c > 0$, such that:

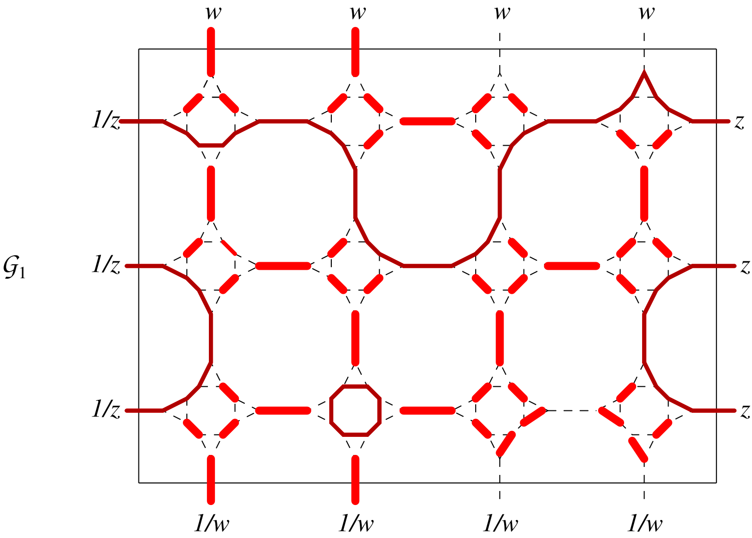
$$\det K_1(z,w) = c \det \Delta_1(z,w).$$

Dimer characteristic polynomial $\det K_1(z,w)$

- contains all information concerning the dimer model on $\mathcal{G}$.
- counts 'double dimer configurations' on $\mathcal{G}$: disjoint unions of
  - trivial cycles of even length and doubled edges,
  - parallel non-trivial cycles with contributions, z, z^{-1}, w, w^{-1},

  which cover all vertices.
- corresponds to 'double Ising configurations' through Fisher's correspondence.

Dimer characteristic polynomial $\det K_1(z, w)$



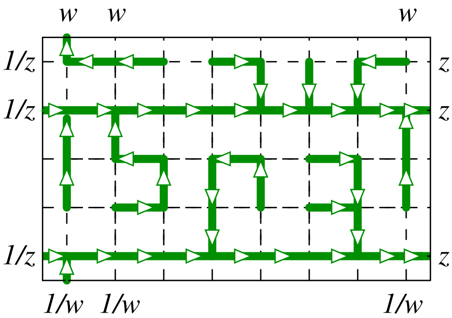
An example of configuration counted by $\det K_1(z, w)$.

Laplacian charact. polynomial $\det \Delta_1(z,w)$

- counts Cycle Rooted Spanning Forests (CRSFs) of G_1: disjoint unions of trees rooted on non-trivial cycles, with weights z, z^{-1}, w, w^{-1}, which cover all vertices.
- Let $\mathcal{F}(G_1)$ denote the set of CRSFs of G_1.

Theorem (Kirchoff, Forman)

$$\det \Delta_1(z,w) = \sum_{F \in \mathcal{F}(G_1)} \left(\prod_{e \in F} \tan \theta_e \right) \prod_{T \in F} (1 - z^{h(T)} w^{v(T)}).$$



Critical Ising model and spanning forests

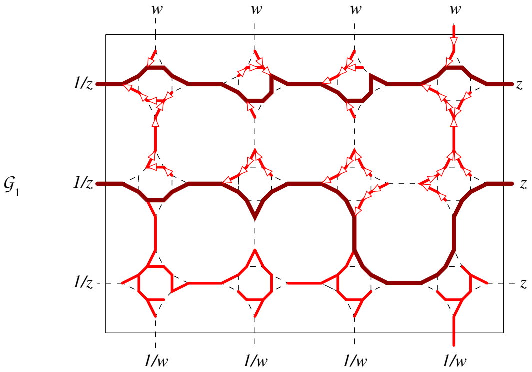
Theorem (dT)

There exists an explicit mapping from CRSFs of G_1 counted by $\det \Delta_1(z, w)$ *to 'double-dimer' configurations of $\mathcal{G}_1$ counted by* $\det K_1(z, w)$.

Explicit mapping

1. "Kirchoff" theorem for the Kasteleyn matrix K:

$$\det \bar{K}_1(z,w) = \sum_{F \in \mathcal{F}(\mathcal{G}_1)} \left(\prod_{e=(x,y)\in F} \bar{K}_{x,y} \right) \prod_{T \in F} (1 - z^{h(T)} w^{v(T)}).$$

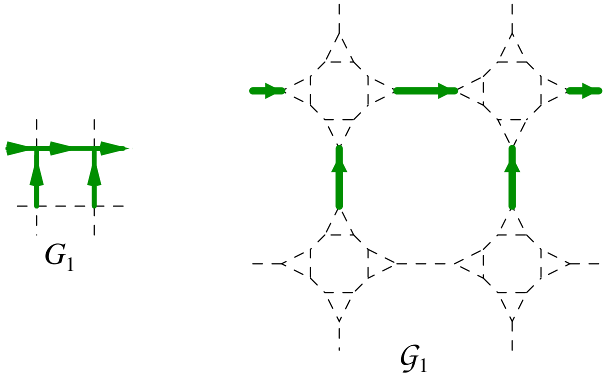


Mapping

2. Characterize CRSFs which contribute to $\det K_1(z,w)$:

3. CRSF $\mathbf{F}$ of $G_1 \to$ Family of CRSFs of $\mathcal{G}_1$, $\mathcal{F}(\mathbf{F})$ which:

- contain exactly the long edges of $\mathbf{F}$,

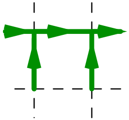


Mapping

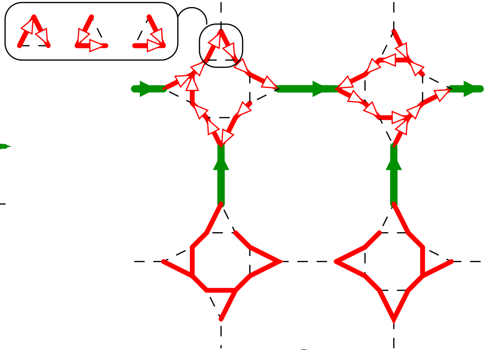
2. Characterize CRSFs which contribute to $\det K_1(z, w)$:

3. CRSF $\mathbf{F}$ of $G_1 \to$ Family of CRSFs of $\mathcal{G}_1$, $\mathcal{F}(\mathbf{F})$ which:

- contain exactly the long edges of $\mathbf{F}$,
- contribute to $\det K_1(z, w)$,



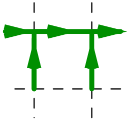
G_1



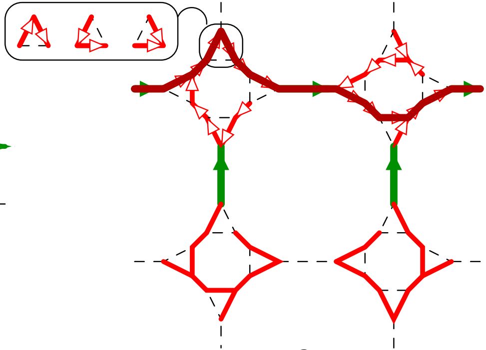
$\mathcal{G}_1$

Mapping

2. Characterize CRSFs which contribute to $\det K_1(z,w)$:
3. CRSF $\mathbf{F}$ of $G_1 \to$ Family of CRSFs of $\mathcal{G}_1$, $\mathcal{F}(\mathbf{F})$ which:
   - contain exactly the long edges of $\mathbf{F}$,
   - contribute to $\det K_1(z,w)$,
   - are in the same homology class as $\mathbf{F}$ in G_1.



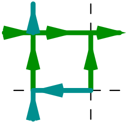
G_1



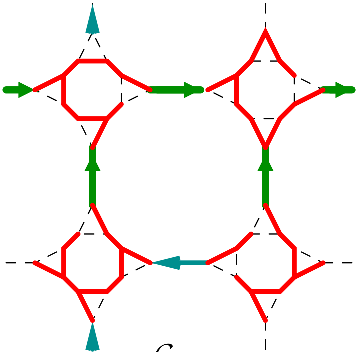
$\mathcal{G}_1$

Mapping

4. Let $\mathbf{E} = \{\mathbf{e}_1, \cdots, \mathbf{e}_k\}$ be a subset of oriented edges of $E(G_1) \setminus \mathbf{F}$
$\rightarrow$ Family of CRSFs of $\mathcal{G}_1$, $\mathcal{F}(\mathbf{F} \cup \mathbf{E})$ which:
   - contain exactly the long edges, $\mathbf{F}, \mathbf{e}_1, \cdots, \mathbf{e}_k$,
   - contribute to $\det K_1(z, w)$,
   - are in the same homology class as $\mathbf{F}$ in G_1.



G_1

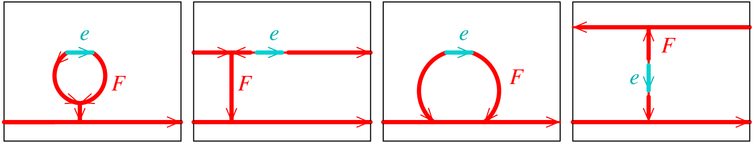


$\mathcal{G}_1$

Mapping

4. Local operation (on a toroidal graph G)
   - CRSF of G is a subset of oriented edges, such that:
     - it contains no trivial cycle,
     - every vertex of G has exactly one outgoing edges of F.
   - Let F be a CRSF of G, e an oriented edge of $E(G_1) \setminus F$

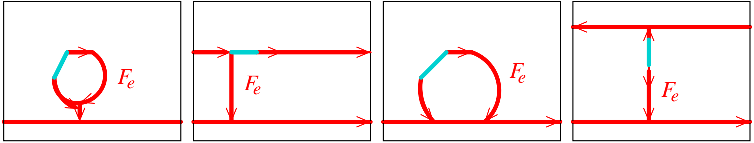
     $\Rightarrow F \cup e$ is one of the following:



Mapping

4. Local operation (on a toroidal graph G)
   - CRSF of G is a subset of oriented edges, such that:
     - it contains no trivial cycle,
     - every vertex of G has exactly one outgoing edges of F.
   - Let F be a CRSF of G, e an oriented edge of $E(G_1) \setminus F$

     $\Rightarrow F \cup e$ is one of the following:

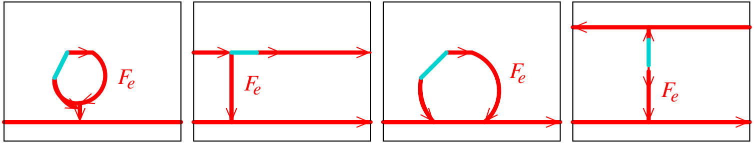


   - $\Rightarrow$ one defines a new CRSF, F_e which contains e.

Mapping

4. Local operation (on a toroidal graph G)
   - CRSF of G is a subset of oriented edges, such that:
     - it contains no trivial cycle,
     - every vertex of G has exactly one outgoing edges of F.
   - Let F be a CRSF of G, e an oriented edge of $E(G_1) \setminus F$

     $\Rightarrow F \cup e$ is one of the following:



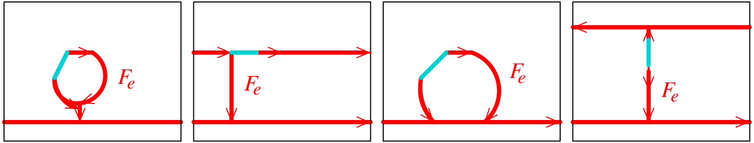
   - $\Rightarrow$ one defines a new CRSF, F_e which contains e.
   - In the boundary cases, one adds trivial cycles, non-trivial cycles...

     $\Rightarrow$ Licit local operations.

Mapping

4. Local operation (on a toroidal graph G)
   - CRSF of G is a subset of oriented edges, such that:
     - it contains no trivial cycle,
     - every vertex of G has exactly one outgoing edges of F.
   - Let F be a CRSF of G, e an oriented edge of $E(G_1) \setminus F$

     $\Rightarrow F \cup e$ is one of the following:



   - $\Rightarrow$ one defines a new CRSF, F_e which contains e.
   - In the boundary cases, one adds trivial cycles, non-trivial cycles...

     $\Rightarrow$ Licit local operations.

- Apply this recursively with $\{\mathbf{e}_1, \cdots, \mathbf{e}_k\}$ on CRSFs F of $\mathcal{F}(\mathbf{F})$.

Mapping

Define

$$\mathcal{S}(\mathbf{F}) = \bigcup_{\mathbf{E} \subset E(G_1) \setminus \mathbf{F}} \mathcal{F}(\mathbf{F} \cup \mathbf{E})$$

$$\mathcal{S} = \bigcup_{\mathbf{F}, \text{CRSFs of } G_1} \mathcal{S}(\mathbf{F}).$$

Theorem

- *For every* $\mathbf{F}_1 \neq \mathbf{F}_2$, $\mathcal{S}(\mathbf{F}_1) \cap \mathcal{S}(\mathbf{F}_2) = \emptyset$,
- $\mathcal{S}$ *contains exactly the CRSFs which contribute to* $\det K_1(z, w)$.
- *The mapping is weight preserving.*