Advertisement – Program on Conformal Geometry

- Where? – Simons Center, Stony Brook
- When? – January–April 2013
- Organizers: Ilia Binder, John Cardy, Andrei Okounkov, Paul Wiegmann
- Workshops:
  - “Integrable structures in random processes" January 21–25 (Binder-Okounkov)
  - “Random Tiling" February 11–15 (Kenyon-de Gier -Nienhuis)
  - “Conformal invariance in continuous and discrete systems" April 8–12 (Cardy-Wiegmann-Lawler)
- for more details visit
  `http://scgp.stonybrook.edu/scientific/programs/`
  or contact one of the organizers

Holomorphic Parafermions on the Lattice and in Conformal Field Theory

John Cardy

University of Oxford

Helsinki, June 2012

Outline

- some pre-history
- discretely holomorphic parafermions in Ising and $\mathbb{Z}_N$ models
- discretely holomorphic parafermions from random curves
- discrete holomorphicity and integrability
- holomorphic parafermions in CFT and SLE

Some Prehistory

- Green [1953] invented the term parastatistics to describe multi-particle states in quantum field theory which transform according to non-trivial representations of S_n
- Fradkin-Kadanoff [1980] identified parafermions in classical 2d lattice models which were generalisations of Kaufman's 1949 Ising model fermions: they argued that in the scaling limit at criticality

$$\langle \psi_\sigma(z_1)\psi_\sigma(z_2)\rangle \propto (z_1 - z_2)^{-2\Delta}(z_1^* - z_2^*)^{-2\overline{\Delta}}$$

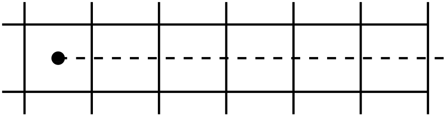
- $\sigma = \Delta - \overline{\Delta}$ is 'fractional spin'
- if Δ or $\overline{\Delta} = 0$, ψ_σ is (anti-)holomorphic
- in 2d quantum models (eg for the fractional quantum Hall effect) these were called anyons [Wilczek, 1982]

Example: the Ising model

- $\mathbb{Z}^2$ lattice, degrees of freedom $s_r = \pm 1$, weights

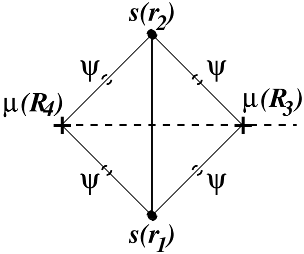
$$e^{\sum_{rr'} J_{rr'} s(r)s(r')} \propto \prod_{rr'} \left(1 + x_{rr'} s(r)s(r')\right)$$

- $\mathbb{Z}_2$ symmetry under $s(r) \to -s(r)$
- disorder variable $\mu(R)$: $s(r)$ identified with $-s(r)$ as r goes around R
- equivalent to taking $x_{rr'} \to -x_{rr'}$ on edges (rr') which cross a 'string' attached to R:



- define fermion $\psi_\sigma(rR)$ on the edge (rR):

$$\psi_\sigma(rR) = s(r) \cdot \mu(R)\, e^{-i\sigma\theta_{rR}}$$



$$\mu(R_4) = \frac{1 - x\, s(r_1)s(r_2)}{1 + x\, s(r_1)s(r_2)}\, \mu(R_3)$$

$$\left(1 + xs(r_1)s(r_2)\right)\mu(R_4) = \left(1 - xs(r_1)s(r_2)\right)\mu(R_3)$$

- multiply both sides by $s(r_1)$ and $s(r_2)$ and use $s^2 = 1$:

$$\begin{aligned} s(r_1)\mu(R_4) + xs(r_2)\mu(R_4) &= s(r_1)\mu(R_3) - xs(r_2)\mu(R_3) \\ xs(r_1)\mu(R_4) + s(r_2)\mu(R_4) &= -xs(r_1)\mu(R_3) + s(r_2)\mu(R_3) \end{aligned}$$

- on the other hand we have discrete holomorphicity if
$$e^{i\pi/4}\psi_{13} + e^{3i\pi/4}\psi_{23} + e^{5i\pi/4}\psi_{24} + e^{7i\pi/4}\psi_{14} = 0$$

- these are consistent if $\sigma = \frac{1}{2}$ and $x = \tanh J = \sqrt{2} - 1$ - the condition for the isotropic Ising model to be critical!

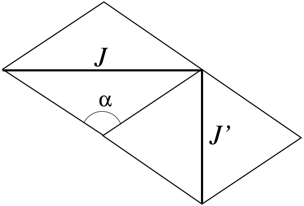
$$\left(1 + xs(r_1)s(r_2)\right)\mu(R_4) = \left(1 - xs(r_1)s(r_2)\right)\mu(R_3)$$

- multiply both sides by $s(r_1)$ and $s(r_2)$ and use $s^2 = 1$:

$$\begin{aligned} s(r_1)\mu(R_4) + xs(r_2)\mu(R_4) &= s(r_1)\mu(R_3) - xs(r_2)\mu(R_3) \\ xs(r_1)\mu(R_4) + s(r_2)\mu(R_4) &= -xs(r_1)\mu(R_3) + s(r_2)\mu(R_3) \end{aligned}$$

- on the other hand we have discrete holomorphicity if
$$e^{i\pi/4}\psi_{13} + e^{3i\pi/4}\psi_{23} + e^{5i\pi/4}\psi_{24} + e^{7i\pi/4}\psi_{14} = 0$$
- these are consistent if $\sigma = \frac{1}{2}$ and $x = \tanh J = \sqrt{2} - 1$ - the condition for the isotropic Ising model to be critical!
- this is true inside all correlation functions with other observables elsewhere and with all (compatible) boundary conditions

Anisotropic case



- more generally we can take different couplings J, J' in different directions
- we then get discrete holomorphicity only if we embed the lattice in $\mathbb{R}^2$ so each face is a rhombus of interior angle α and

$$\tanh J = \tan((\pi - \alpha)/4), \qquad \tanh J' = \tan(\alpha/4)$$

which implies $\sinh 2J \sinh 2J' = 1$, the condition for criticality

$\mathbb{Z}_N$ models

- more generally we can take $s(r) \in (1, \omega, \ldots, \omega^{N-1})$ where $\omega = e^{2\pi i/N}$
- nearest neighbour interaction with most general weights

$$\prod_{rr'} \left(1 + \sum_{j=1}^{N-1} w_j \big(s(r)^* s(r')\big)^j + \text{c.c.} \right)$$

and similarly w'_j on the vertical edges

- define disorder variables $\mu(R)$ by $s \to \omega s$ across the string
- define parafermions

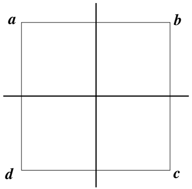
$$\psi_\sigma \equiv e^{-i\sigma\theta_{rR}} s(r) \mu(R)$$

- these are discretely holomorphic [Rajabpour, JC 2008] if $\sigma = (N-1)/N$ and

$$w_j = x_j(\alpha) = \prod_{j'=0}^{j-1} \frac{\sin(2\pi j' + \alpha)/2N)}{\sin(2\pi(j'+1) - \alpha)/2N)}, \qquad w'_j = x_j(\pi - \alpha)$$

- these are the weights of the Fateev-Zamolodchikov model [1982], which is critical and integrable in the sense of Yang-Baxter
- F-Z [1985] assumed these parafermions are holomorphic in the scaling limit and used them as building blocks of the corresponding CFT

Yang-Baxter Equations

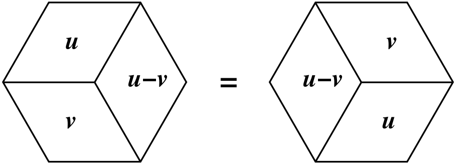


- consider a planar lattice built out of elementary faces or plaquettes, e.g. $\mathbb{Z}^2$
- the degrees of freedom $(a, b, \cdots)$ live on the vertices and the weight for a given configuration is

$$\propto \prod_{\text{faces}} W(a, b, c, d)$$

- such models can also be realised as vertex models with degrees of freedom on the edges

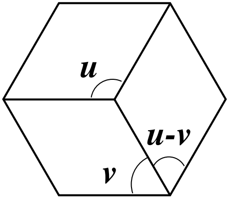
- the weights $W(a,b,c,d;u)$ depend on a real variable u (spectral parameter) such that, when summed over the central degree of freedom c



$$\sum_c W(.,.,c,.;u)W(.,c.,.,.;v)W(.,.,.,c;u-v)$$

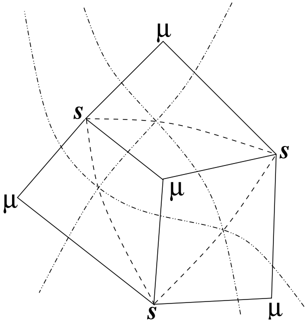
$$= \sum_c W(.,c,.,.;u-v)W(.,.,.,c;v)W(c,.,.,.;u)$$

- for Ising and $\mathbb{Z}_N$ models $W(a,b,c,d;u) \propto w(a,c;u)w(b,d;u')$ and Y-B equations are usually called star-triangle relations

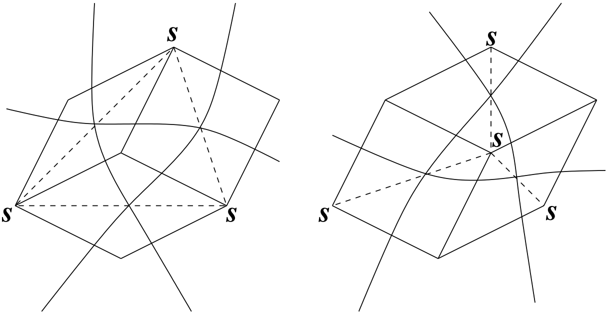


- the discretely holomorphic weights of the Ising and Z_N models satisfy the Yang-Baxter equations with α identified as the spectral parameter u
- the spectral parameter tells us how to embed the lattice in $\mathbb{R}^2$ so as to get discrete holomorphicity (and conjecturally full conformal invariance in the scaling limit)

More general lattices



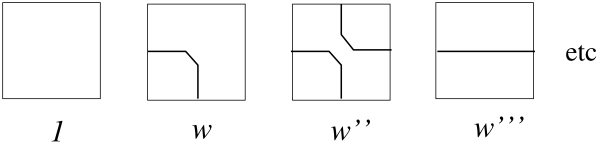
- this extends to an inhomogeneous 2-colourable ‘Baxter lattice’: such a lattice can always be embedded in $\mathbb{R}^2$ so that all its faces are rhombi, so it is isoradial: if the local weights are those determined by the local angle α then ψ_σ is discretely holomorphic

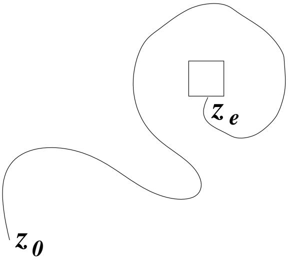


- the Yang-Baxter equations are equivalent to saying that different tilings of a hexagon do not change measure in exterior

Discretely holomorphic observables and curves

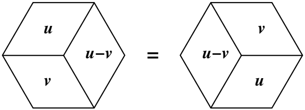
- many (but not all?) interesting lattice models may also be realised in terms of non-intersecting curves
- Smirnov showed how to construct discretely holomorphic observables for some simple examples
- a more instructive example is the $O(n)$ model on $\mathbb{Z}^2$ lattice [Nienhuis,Blöte]



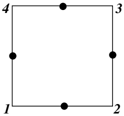


- in this case one considers curves $\gamma_{z_0 z_e}$ from some point z_0 ending at a given edge z_e, with turning angle $\theta_{z_0 z_e}$
- $\psi_\sigma(z_e) \equiv \mathbb{E}[e^{-i\sigma\theta_{z_0 z_e}}]$
- $\sum_\square \psi_\sigma(z_e)\delta z_e = 0$ if the weights satisfy the critical Y-B equations and the faces are embedded in $\mathbb{R}^2$ as rhombi with angle $\alpha = u$ [Ikhlef, JC 2009]

- Yang-Baxter equations are cubic functional equations for $W(\cdots;u)$

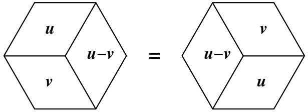


- discrete holomorphicity is a linear condition on $W(\cdots;u)$ for a fixed u

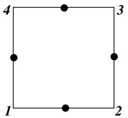


Discrete Holomorphicity and Yang-Baxter

- Yang-Baxter equations are cubic functional equations for $W(\cdots;u)$

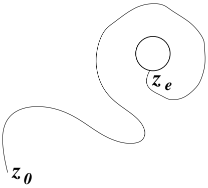


- discrete holomorphicity is a linear condition on $W(\cdots;u)$ for a fixed u



- does one imply the other in general? Are there counter-examples?
- connection is clearer in the limit $u-v\to 0$ when rhombus degenerates and both sets of equations simplify

Holomorphic parafermions from the continuum



- curve from z_0 to point on boundary of disc $z_e = z + \epsilon e^{i\theta}$
- in CFT language consider [cf. Werness' SLE approach]

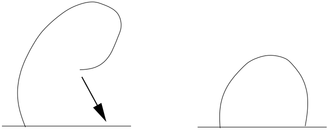
$$\psi_\sigma(z) \stackrel{?}{=} \lim_{\epsilon \to 0} \int d\theta e^{-i\sigma\theta} \phi_{\text{boundary}}(z + \epsilon e^{i\theta})$$

- [Simmons + JC] for $\sigma = \Delta_{21} = (6 - \kappa)/2\kappa$ limit exists and $\psi_\sigma(z)$ is holomorphic
- moreover its correlators satisfy 2nd order linear differential equations wrt z

Holomorphic parafermions and CFT

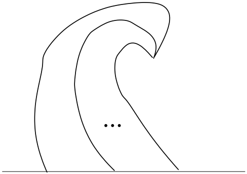
- in the half-plane the parafermionic observable corresponds to the CFT correlation function

$$\langle\phi_{21}(z_0)\psi_\sigma(z)\rangle_{\mathbb{H}} \sim (z_0 - z)^{-\Delta_{21}-\sigma}$$



- as $z \to$ boundary, $\psi_\sigma \to \phi_{21}$
- conformal invariance implies that these have the same scaling dimension and hence $\sigma = \Delta_{21} = (6-\kappa)/2\kappa$

- extension to N curves



- this suggests that bulk holomorphic parafermions exist with $\sigma = \Delta_{N+1,1}$
- $N = 2$ is already identified in terms of boundary curves of F-K clusters [Smirnov, Riva+JC]
- in CFT every boundary correlation function has a an extension into the complex z-plane – this suggests that to each boundary operator in a given CFT with scaling dimension Δ , there exists a holomorphic bulk operator with conformal spin Δ
- if so, there may be a lot more possible discretely holomorphic observables out there!

Some outstanding problems

- is it always true that [cf Fendley's talk]

  discrete holomorphicity ⇔ criticality + Yang-Baxter integrability?

  - can one do something useful with this? [eg Smirnov–Duminil-Copin]
  - boundary extensions [see Guttmann, Ikhlef talks]

- can convergence of ψ_σ to a truly holomorphic quantity be proved for cases other than the Ising model?
- major problem: not enough equations!
  - can this be used to prove convergence of the measure on curves to SLE à la Smirnov?
  - can this be used as a constructive route to the full scaling theory/CFT? [cf. talks by Hongler, Chelkak, Izyurov]

Advertisement – Program on Conformal Geometry

- Where? – Simons Center, Stony Brook
- When? – January–April 2013
- Organizers: Ilia Binder, John Cardy, Andrei Okounkov, Paul Wiegmann
- Workshops:
  - “Integrable structures in random processes" January 21–25 (Binder-Okounkov)
  - “Random Tiling" February 11–15 (Kenyon-de Gier -Nienhuis)
  - “Conformal invariance in continuous and discrete systems" April 8–12 (Cardy-Wiegmann-Lawler)
- for more details visit `http://scgp.stonybrook.edu/scientific/programs/` or contact one of the organizers