TOPICS IN GEOMETRIC FOURIER ANALYSIS

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Contents

These notes are quite inofficial and meant to give some support for the lectures. They are not complete, for example, in the lectures more details of the proofs can be given.

The lecture notes of Wolff [W] is the book closest to these lectures in content and spirit. Stein's book [S] covers also large part of this material. Some topics are discussed in Sogge's book [So] and those of Grafakos [G1] and [G2]. The lecture notes of Mitsis [Mi] give a nice survey. For the basics of Fourier analysis Duoandikoetxea [D] is excellent.

1. Fourier transforms

1.1. Fourier transform in L^1 and L^2. The following basic facts about Fourier transforms of functions can be found in most standard books in Fourier analysis. Good references are [D] and [W].

The Fourier transform of a Lebesgue integrable function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int f(x)e^{-2\pi i\xi\cdot x}dx. \tag{1.1}$$

The following formulas follow easily by Fubini's theorem:

$$\int \hat{f}g = \int f\hat{g}, f, g \in L^1, \text{ (product formula)}, \tag{1.2}$$

$$\widehat{(f * g)} = \hat{f}\hat{g}, f, g \in L^1, \text{ (convolution formula)}. \tag{1.3}$$

Trivial changes of variables show how Fourier transform behaves under simple transformations. For $a \in R^n$ and $r > 0$ define the translation τ_a and dilation δ_r by

$$\tau_a(x) = x + a,\ \delta_r(x) = rx.$$

Then for $f \in L^1$,

$$\widehat{f \circ \tau_a}(\xi) = e^{2\pi i a\cdot\xi}\hat{f}(\xi),\ \mathcal{F}(e^{2\pi i a\cdot x}f)(\xi) = \hat{f}(\xi - a), \tag{1.4}$$

$$\widehat{f \circ \delta_r}(\xi) = r^n\hat{f}(r\xi). \tag{1.5}$$

The orthogonal group $O(n)$ of $\mathbb{R}^n$ consists of linear maps $g : \mathbb{R}^n \to \mathbb{R}^n$ which preserve inner product: $g(x)\cdot g(y) = x\cdot y$. Then

$$\widehat{f \circ g} = \hat{f} \circ g \text{ for } g \in O(n). \tag{1.6}$$

The proof of the following *Riemann-Lebesgue lemma* is also easy:

$$\hat{f}(\xi) \to 0 \text{ when } |\xi| \to \infty \text{ and } f \in L^1. \tag{1.7}$$

The following inversion formula is a bit trickier to prove:

$$f(x) = \int \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \text{ if } f, \hat{f} \in L^1. \tag{1.8}$$

We denote the inverse Fourier transform of $g \in L^1$ by

$$\mathcal{F}^{-1}(g)(x) = \check{g}(x) = \int g(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

Proof. Here is a quick proof, some details more are given in [D] and [W]. Define

$$\Gamma(x) = e^{-\pi |x|^2}, \Gamma_\varepsilon(x) = e^{-\pi \varepsilon^2 |x|^2}.$$

Then $\hat{\Gamma} = \Gamma$. This follows from the definitions by complex integration or by observing that when $n = 1$, Γ and $\hat{\Gamma}$ satisfy the same differential equation $f'(x) = -2\pi x f(x)$ with the initial condition $f(0) = 1$. We have then

$$\hat{\Gamma}_\varepsilon(\xi) = \varepsilon^{-n} e^{-\pi |\xi|^2 / \varepsilon^2}.$$

Denote

$$I_\varepsilon(x) = \int \hat{f}(\xi) e^{-\pi \varepsilon^2 |x|^2} e^{2\pi i \xi \cdot x} d\xi.$$

Then by Lebesgue's dominated convergence theorem,

$$I_\varepsilon(x) \to \int \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \text{ as } \varepsilon \to 0.$$

On the other hand, denoting $g_x(y) = e^{-\pi \varepsilon^2 |y|^2} e^{2\pi i \xi \cdot y}$, we have $\widehat{g_x}(y) = \widehat{\Gamma_\varepsilon}(y - x) = \Gamma^\varepsilon(x - y)$, where $\Gamma^\varepsilon(y) = \varepsilon^{-n} \Gamma(y/\varepsilon)$. By the product formula,

$$I_\varepsilon(x) = \int \hat{f} g_x = \int f \widehat{g_x} = \Gamma^\varepsilon * f(x).$$

The functions $\Gamma^\varepsilon, \varepsilon > 0$, provide a standard approximate identity for which $\Gamma^\varepsilon * f \to f$ as $\varepsilon \to 0$. The combination of these two limits gives the inversion formula. □

The Schwartz space $\mathcal{S}$ of rapidly decreasing functions is very convenient in Fourier analysis. It consists of infinitely differentiable complex valued functions f on $\mathbb{R}^n$ which together with their partial derivatives of all orders tend to zero at infinity more quickly than $|x|^{-k}$ for all integers k. Observe that $C_0^\infty \subset \mathcal{S}$.

The first basic fact is that

$$f \in \mathcal{S} \text{ if and only if } \hat{f} \in \mathcal{S}. \tag{1.9}$$

This follows from the formulas for partial derivatives, which in turn follow easily by partial integration: if $f \in \mathcal{S}$ (or more generally under some obvious conditions):

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi), \tag{1.10}$$

(1.11) $$\partial^\alpha \hat{f}(\xi) = \mathcal{F}((-2\pi i x)^\alpha f)(\xi).$$

Here $\alpha = (\alpha_1, \dots, \alpha_n), \alpha_j \in \mathbb{N} = \{0, 1, \dots\}, x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$ and ∂^α means α_j partial derivatives with respect to x_j.

Secondly, we have

(1.12) $$\int f\bar{g} = \int \hat{f}\bar{\hat{g}}, f, g \in \mathcal{S}, \text{ (Parseval)},$$

(1.13) $$||f||_2 = ||\hat{f}||_2, f, g \in \mathcal{S}, \text{ (Plancherel)},$$

Parseval's formula (which of course gives Plancherel's formula) is an easy consequence of the inversion formula and the product formula:

$$\int f\bar{g} = \int \hat{\hat{f}}(-x)\bar{g}(x)dx = \int \hat{\hat{f}}(x)\bar{g}(-x)dx = \int \hat{f}(x)\hat{h}(x)dx,$$

where $h(x) = \bar{g}(-x)$. We see immediately from the definition of Fourier transform that $\hat{h}(x) = \overline{\hat{g}(x)}$, which proves Parseval's formula.

So the Fourier transform is a linear L^2-isometry of $\mathcal{S}$ onto itself. The formula (1.1) cannot be used to define Fourier transform for L^2-functions; the integral need not exist if f is not integrable. But $\mathcal{S}$ is dense in L^2, so (1.9) and (1.15) give immediately a unique linear extension of the Fourier transform to L^2. Thus we have $\hat{f}$ defined for all $f \in L^1 \cup L^2$. Parseval's and Plancherel's formulas extend now immediately to L^2:

(1.14) $$\int f\bar{g} = \int \hat{f}\bar{\hat{g}}, f, g \in L^2, \text{ (Parseval)},$$

(1.15) $$||f||_2 = ||\hat{f}||_2, f, g \in L^2, \text{ (Plancherel)}.$$

1.2. **Fourier transforms of measures and distributions.** We denote by $\mathcal{M}(\mathbb{R}^n)$ the set of finite Borel measures μ on $\mathbb{R}^n$ (outer measures for which Borel sets are measurable, but sometimes we may mean by Borel measure also a signed or complex measure). The Fourier transform of $\mu \in \mathcal{M}(\mathbb{R}^n)$ is defined by

(1.16) $$\hat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu x.$$

When μ has compact support, $\hat{\mu}$ is a bounded Lipschitz continuous function (an easy exercise). It need not be in any L^p for $p < \infty$; for example $\hat{\delta}_a(\xi) = e^{-2\pi i \xi \cdot a}$. The product formula has by Fubini's theorem an easy extension to measures:

(1.17) $$\int \hat{\mu} d\nu = \int \hat{\nu} d\mu, \mu, \nu \in \mathcal{M}(\mathbb{R}^n).$$

As for functions, we can approximate measures with smooth compactly supported functions using convolution. Let (ψ_ε) be a C^∞ approximate identity such that

$$\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon),\ \varepsilon > 0,\ \psi \geq 0,\ \int \psi = 1.$$

Then

$$\hat{\psi}_\varepsilon(\xi) = \hat{\psi}(\varepsilon\xi) \to \psi(0) = \int \psi = 1 \text{ as } \varepsilon \to 0.$$

We define the the convolution $f * \mu$ by

$$f * \mu(x) = \int f(x-y)d\mu y,$$

when the integral exists. Setting for a finite Borel measure μ, $\mu_\varepsilon = \psi_\varepsilon * \mu$, we have that μ_ε converges weakly to μ as $\varepsilon \to 0$, that is,

$$\int g\mu_\varepsilon \to \int g d\mu \text{ for all } g \in C_0(\mathbb{R}^n),$$

and

$$\widehat{\mu_\varepsilon} = \hat{\psi}_\varepsilon \hat{\mu} \to \hat{\mu} \text{ uniformly}.$$

We have for $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $f \in \mathcal{S}$ (and for much more general f),

$$\widehat{f * \mu} = \hat{f}\hat{\mu}, \tag{1.18}$$

$$\widehat{f\check{\mu}} = \hat{f} * \mu, \tag{1.19}$$

$$\widehat{f\mu} = \hat{f} * \hat{\mu}. \tag{1.20}$$

We leave the easy proofs as exercises.

A very general way to define Fourier transform is to do it for distributions:

Definition 1.1. A tempered distribution is a continuous (in a suitable sense, see [D]) linear functional $T : \mathcal{S} \to \mathbb{C}$. Its Fourier transform is the tempered distribution $\hat{T}$ defined by

$$\hat{T}(\varphi) = T(\hat{\varphi}) \text{ for } \varphi \in \mathcal{S}.$$

This definition agrees with the earlier ones when T corresponds to a function in $L^1 \cup L^2$ or a finite Borel measure.

1.3. **Fourier transform in L^p.** All L^p-functions, $1 \leq p \leq \infty$, and more generally all locally integrable functions f such that for some constants C and m, $|f(x)| < C|x|^m$ when $|x| > 1$, can be considered as tempered distributions T_f:

$$T_f(\varphi) = \int f\varphi, \varphi \in \mathcal{S},$$

and so they have Fourier transform as a tempered distribution.

For $1 < p < 2$ we can also make use of L^1 and L^2: any $f \in L^p, 1 < p < 2$, can be written as $f = f_1 + f_2, f_1 \in L^1, f_2 \in L^2$, and then $\hat{f} = \widehat{f_1} + \widehat{f_2}$. For $p = 2$ we have the Plancherel identity and for $p = 1$ we have the trivial estimate:

$$\|\hat{f}\|_\infty \leq \|f\|_1.$$

From these the Riesz-Thorin interpolation theorem gives the following Hausdorff-Young inequality:

$$\|\hat{f}\|_q \leq \|f\|_p \text{ for } f \in L^p, 1 < p < 2, q = \frac{p}{p-1}. \tag{1.21}$$

No such inequality holds when $p > 2$.

Since we shall use Riesz-Thorin also later, we state it here. For a proof, see for example [K] or [G1].

Theorem 1.2. Let (X, μ) and (Y, ν) be two measure spaces and T a linear operator such that, for some $1 \leq p_0, p_1, q_0, q_1 \leq \infty$,

$$\|T(f)\|_{L^{q_0}(\nu)} \leq C_0 \|f\|_{L^{p_0}(\mu)} \text{ for all } f \in L^{p_0}(\mu),$$

and

$$\|T(f)\|_{L^{q_1}(\nu)} \leq C_1 \|f\|_{L^{p_1}(\mu)} \text{ for all } f \in L^{p_1}(\mu),$$

Then for all $0 < \theta < 1$,

$$\|T(f)\|_{L^q(\nu)} \leq C_0^{1-\theta} C_1^\theta \|f\|_{L^p(\mu)} \text{ for all } f \in L^p(\mu),$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

2. Fourier transform of radial functions

This section is mainly based on [SW]. Watson's book [Wa] contains a lot of information on Bessel functions.

One of the goals of this section is to find the Fourier transform of the surface measure on the sphere

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

Let's first compute the simpler example of the length measure λ on the line segment $I = [-(1,0), (1,0)]$ in $\mathbb{R}^2$:

$$\hat{\lambda}(\eta, \xi) = \int_{-1}^{1} e^{-2\pi i(\eta x + \xi 0)} dx = \int_{-1}^{1} \cos(2\pi\eta x) dx = \frac{\sin(2\pi\eta)}{\pi\eta}.$$

We see that $\hat{\lambda}(\eta, \xi)$ tends to 0 for a fixed ξ when η tends to ∞, but it remains constant for a fixed η when ξ tends to ∞, and hence does not tend to 0 when $|(\eta, \xi)| \to \infty$.

We denote by σ^{n-1} the surface measure on S^{n-1}, and sometimes also by σ^m the surface measure on m-dimensional spheres in $\mathbb{R}^n$. Letting $\mathcal{L}^n$ denote the Lebesgue measure, we have that σ^{n-1} is the weak limit of the measures $\delta^{-1}\mathcal{L}^n \lfloor (B(0, 1+\delta) \setminus B(0,1))$ as $\delta \to 0$. Here $\mu \lfloor A$ is the restriction of the measure μ to the set A; $\mu \lfloor A(B) = \mu(A \cap B)$.

To find the Fourier transform of σ^{n-1} we first compute the Fourier transform of a radial function. Suppose $f \in L^1(\mathbb{R}^n), f(x) = \psi(|x|)$, $x \in \mathbb{R}^n$, for some $\psi\colon [0,\infty) \to \mathbb{C}$. We shall use the following two Fubini-type formulas which can either be proven by standard calculus or deduced from a general coarea formula.

If $f \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f\, d\mathcal{L}^n = \int_{S^{n-1}} \left(\int_0^\infty f(rx) r^{n-1}\, dr \right) d\sigma^{n-1} x. \tag{2.1}$$

Fix $e \in S^{n-1}$ and denote $S_\theta = \{x \in S^{n-1} : e \cdot x = \cos\theta\}$. Then for $g \in L^1(S^{n-1})$,

$$\int_{S^{n-1}} g\, d\sigma^{n-1} = \int_0^\pi \left(\int_{S_\theta} g(x)\, d\sigma^{n-2} x \right) d\theta. \tag{2.2}$$

Applying (2.1) and Fubini's theorem,

$$\hat{f}(re) = \int f(y) e^{-2\pi i r e\cdot y} d\mathcal{L}^n y = \int_0^\infty \psi(s) s^{n-1} \left(\int_{S^{n-1}} e^{-2\pi i r s e\cdot x} d\sigma^{n-1} x \right) ds.$$

The inside integral can be computed with the help of (2.2), since $e^{-2\pi i r s e\cdot x}$ is constant in S_θ:

$$\int_{S^{n-1}} e^{-2\pi i r s e\cdot x} d\sigma^{n-1} x = \int_0^\pi e^{-2\pi i r s \cos\theta} \sigma^{n-2}(S_\theta)\, d\theta.$$

The set S_θ is an $(n-2)$-dimensional sphere of radius $\sin\theta$, so

$$\sigma^{n-2}(S_\theta) = b(n)(\sin\theta)^{n-2},$$

where $b(n) = \sigma^{n-2}(S^{n-2})$.

Changing variable $\cos\theta \mapsto -t$ and introducing the *Bessel functions* $J_m\colon [0,\infty) \to \mathbb{C}$, where $m > -1/2$:

$$J_m(u) := \frac{(u/2)^m}{\Gamma(m+1/2)\Gamma(1/2)} \int_{-1}^1 e^{iut}(1-t^2)^{m-1/2} dt, \tag{2.3}$$

we obtain

$$\begin{aligned}\int_{S^{n-1}} e^{-2\pi i r s e\cdot x} d\sigma^{n-1}(x) &= b(n) \int_{-1}^1 e^{2\pi i r s t}(1-t^2)^{(n-3)/2} dt \\ &= c(n)(rs)^{-(n-2)/2} J_{(n-2)/2}(2\pi r s).\end{aligned}$$

This leads to the formula for the Fourier transform of the radial function f:

$$\hat{f}(x) = c(n)|x|^{-(n-2)/2} \int_0^\infty \psi(s) J_{(n-2)/2}(2\pi|x|s) s^{n/2}\, ds. \tag{2.4}$$

A basic property of Bessel functions is the following decay estimate, which we shall prove in the next section:

$$J_m(t) \le C(m) t^{-1/2} \text{ for } t > 0. \tag{2.5}$$

When $m = k - 1/2, k \in \{1, 2, \dots\}$, repeated partial integrations show that the Bessel function J_m can be written in terms of elementary functions in the form from which (2.5) easily follows. In particular,

$$J_{1/2}(t) = \frac{\sqrt{2}}{\sqrt{\pi t}} \sin t. \tag{2.6}$$

All Bessel functions behave roughly like this at infinity, that is,

$$J_m(t) = \frac{\sqrt{2}}{\sqrt{t}} \cos(t - \pi m/2 - \pi/4) + O(t^{-3/2}),\ t \to \infty. \tag{2.7}$$

This can be verified with a fairly simple integration, see [SW], pp. 158-159.

Applying the formula (2.4) to the characteristic function of the annulus $B(0, 1 + \delta) \setminus B(0, 1)$ and letting $\delta \to 0$, we get

$$\widehat{\sigma^{n-1}}(x) = c(n)|x|^{(2-n)/2} J_{(n-2)/2}(2\pi|x|). \tag{2.8}$$

Consequently,

$$|\widehat{\sigma^{n-1}}(x)| \leq C(n, m)|x|^{(1-n)/2} \text{ for } x \in \mathbb{R}^n. \tag{2.9}$$

This is the best possible decay any measure on a smooth hypersurface, in fact, on any set of Hausdorff dimension $n - 1$. The reason for getting such a good decay for $\widehat{\sigma^{n-1}}$ is curvature; for example segments are not curving at all but circles are curving uniformly. Also for more general surfaces curvature properties play a central role in the behaviour of Fourier transforms. We shall discuss this a little more in the next section.

3. Estimates on oscillatory integrals (stationary phase)

Sogge [So] and Wolff [W] cover this material and Stein [S] goes much further.

In this section we study integrals of the type

$$I(\lambda) = \int e^{i\lambda\varphi(x)} \psi(x) dx,\ \lambda > 0, \tag{3.1}$$

and in particular their behaviour as $\lambda \to \infty$. As a standing assumption the functions φ and ψ defined on $\mathbb{R}^n$ will be smooth and ψ will have compact support, φ is real valued and ψ complex valued. As special cases we obtain the estimates for the Bessel functions and Fourier transform of the surface measure on the sphere mentioned in the previous section.

3.1. One-dimensional case.

We begin by studying the one-dimensional case.

Theorem 3.1. If $\varphi'(x) \neq 0$ when $x \in \operatorname{spt} \psi$, then for every $N \in \mathbb{N}$,

$$I(\lambda) \leq C_N \lambda^{-N}.$$

The constant C_N may of course depend on φ and ψ (here and later in this section). In particular, we can take

$$C_1 = \int |\frac{d}{dx}(\frac{\psi(x)}{\varphi'(x)})|dx.$$

Proof. Integrating by parts,

$$|I(\lambda)| = |\int \frac{1}{i\lambda\varphi'(x)}\frac{d}{dx}(e^{i\lambda\varphi(x)})\psi(x)dx| = |-\int e^{i\lambda\varphi(x)}\frac{d}{dx}(\frac{\psi(x)}{i\lambda\varphi'(x)})dx| \leq C_1/\lambda.$$

The cases $N \geq 2$ follow by similar calculations. □

If $\varphi'(x) = 0$ but some higher order derivative does not vanish, the following *van der Corput's lemma* is useful:

Theorem 3.2. Suppose $k \in \{1, 2, \dots\}$ is such that $|\varphi^{(k)}(x)| \geq 1$ for $x \in [a, b]$. Then with $C_k = 5 \cdot 2^{k-1} - 2$,

$$|\int_a^b e^{i\lambda\varphi(x)}dx| \leq C_k\lambda^{-1/k} \tag{3.2}$$

if i) $k \geq 2$ or
ii) $k = 1$ and φ' is monotone.

Proof. Suppose first ii). Integrating by parts

$$|\int_a^b e^{i\lambda\varphi(x)}dx| = |\frac{e^{i\lambda\varphi(b)}}{i\lambda\varphi'(b)} - \frac{e^{i\lambda\varphi(a)}}{i\lambda\varphi'(a)} - \int_a^b e^{i\lambda\varphi(x)}\frac{d}{dx}(\frac{1}{i\lambda\varphi'(x)})dx|$$
$$\leq 2\lambda^{-1} + \lambda^{-1}\int_a^b |\frac{d}{dx}(\frac{1}{\varphi'(x)})|dx| = 2\lambda^{-1} + \lambda^{-1}|\varphi'(b)^{-1} - \varphi'(a)^{-1}| \leq 3\lambda^{-1}.$$

where in the last equality and inequality we used the facts that $\frac{d}{dx}(\frac{1}{\varphi'(x)})$ and $\varphi'(x)$ do not change sign on $[a, b]$.

Suppose then that $k \geq 2$. We use induction on k and assume that (3.2) holds for $k-1$. We may assume that $\varphi^{(k)}(x) \geq 1$ for $x \in [a, b]$, since $\varphi^{(k)}$ does not change sign on $[a, b]$. Then $\varphi^{(k-1)}$ is strictly increasing and there is a unique $c \in [a, b]$ such that $|\varphi^{(k-1)}(x)|$ has its minimum at c. Either $\varphi^{(k-1)}(c) = 0$ or $c = a$ or $c = b$. Suppose $\varphi^{(k-1)}(c) = 0$ and let $\delta > 0$. Then $|\varphi^{(k-1)}(x)| \geq \delta$ when $x \in [a, b] \setminus [c - \delta, c + \delta]$ and the induction hypothesis gives

$$|\int_a^{c-\delta} e^{i\lambda\varphi(x)}dx| \leq C_{k-1}(\lambda\delta)^{-1/(k-1)}$$

and

$$|\int_{c+\delta}^b e^{i\lambda\varphi(x)}dx| \leq C_{k-1}(\lambda\delta)^{-1/(k-1)}.$$

Since

$$|\int_{c-\delta}^{c+\delta} e^{i\lambda\varphi(x)}dx| \leq 2\delta,$$

we obtain

$$|\int_a^b e^{i\lambda\varphi(x)}dx| \leq 2C_{k-1}(\lambda\delta)^{-1/(k-1)} + 2\delta.$$

Choosing $\delta = \lambda^{-1/k}$ we get

$$|\int_a^b e^{i\lambda\varphi(x)}dx| \leq (2C_{k-1} + 2)\lambda^{-1/k}.$$

If $c = a$ or $c = b$, a similar argument gives

$$|\int_a^b e^{i\lambda\varphi(x)}dx| \leq C_{k-1}(\lambda\delta)^{-1/(k-1)} + \delta,$$

and we can again take $\delta = \lambda^{-1/k}$.

As $2C_{k-1} + 2 = 5 \cdot 2^{k-1} - 4 < C_k$, the proof is complete. □

Corollary 3.3. Under the assumptions of Theorem 3.2, for any C^∞-function $\psi : \mathbb{R} \to \mathbb{C}$ and for $a < b$,

$$|\int_a^b e^{i\lambda\varphi(x)}\psi(x)dx| \leq C_k\lambda^{-1/k}(|\psi(b)| + \int_a^b |\psi'(x)|dx).$$

Proof. Let

$$F(x) = \int_a^x e^{i\lambda\varphi(t)}dt.$$

Then

$$\int_a^b e^{i\lambda\varphi(x)}\psi(x)dx = \int_a^b F'(x)\psi(x)dx = F(b)\psi(b) - \int_a^b F(x)\psi'(x)dx,$$

and by Theorem 3.2, $|F(x)| \leq C_k\lambda^{-1/k}$ for all $x \in [a, b]$, from which the theorem follows. □

We discuss now applications to Bessel functions and the surface measure σ^{n-1} on the sphere S^{n-1}. We defined in (2.3)

$$J_m(t) := \frac{(t/2)^m}{\Gamma(m + 1/2)\Gamma(1/2)} \int_{-1}^1 e^{its}(1 - s^2)^{m-1/2}ds,$$

for $m > -1/2$. For the formula for radial functions and $\widehat{\sigma^{n-1}}$ we only need the integral and half integral values of m. When $m - 1/2$ is a positive integer, we already observed in (2.7) that the estimate (2.5) holds. When $m \in \mathbb{N}$, we have the alternative formula:

$$J_m(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\sin\theta}e^{-im\theta}d\theta.$$

This is easily checked for $m = 0$, and for $m > 0$ it follows by induction from the following recursion relation, whose proof is a routine verification:

$$\frac{d}{dt}(t^{-m}J_m(t)) = t^m J_{m+1}(t).$$

This alternative formula combined with Corollary 3.3 yields (2.5). We apply Corollary 3.3 with $\lambda = t$ and $\varphi(x) = \sin x$. Then $\varphi'(x) = 0$ when x is $\pi/2$ or $3\pi/2$ and $\varphi''(x) = \pm 1$ for these values of x. We can find functions ψ_1, ψ_2 and ψ_3 such that $\psi_1 + \psi_2 + \psi_3 = 1$, ψ_1 has support and equals 1 near $\pi/2$, and ψ_2 has support and equals 1 near $3\pi/2$. Then we can write $J_m(t)$ as a sum of three terms, to two of them we apply Corollary 3.3 with $k = 2$ and to one of them we apply Theorem 3.1.

Thus we get decay estimate (2.8) for the spherical surface measure. This argument is heavily based on the radial symmetry of the sphere; it helped us to reduce to one-dimensional integrals. For other surface measures we need analogous estimates for higher dimensional integrals, which we investigate now.

3.2. **Higher dimensional case.** For the rest of this section φ and ψ will be smooth functions in $\mathbb{R}^n$ with ψ having a compact support, φ is real valued and ψ complex valued. We denote again

$$I(\lambda) = \int e^{i\lambda\varphi(x)}\psi(x)dx,\ \lambda > 0.$$

Theorem 3.4. If $\nabla\varphi(x) \neq 0$ when $x \in \operatorname{spt}\psi$, then for every $N \in \mathbb{N}$,

$$I(\lambda) \lesssim_N \lambda^{-N}. \tag{3.3}$$

Proof. Suppose first that for some j, $\partial_j\varphi(x) \neq 0$ for $x \in \operatorname{spt}\psi$. Then by Fubini's theorem, writing $\tilde{x} = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, and $C = \{\tilde{x} : x \in \operatorname{spt}\psi\}$,

$$I(\lambda) = \int_C (\int_{\mathbb{R}} e^{i\lambda\varphi(x)}\psi(x)dx_j)d\tilde{x}.$$

An application of Theorem 3.1 to the inner integral yields (3.3), obviously the proof of Theorem 3.1 shows that the constants involved are independenof $\tilde{x}$.

In the general case we can cover $\operatorname{spt}\psi$ with finitely many balls B_k such that some $\partial\varphi_{j_k}(x) \neq 0$ for $x \in B_k$. Writing $\psi = \sum_k \psi_k$ with $\operatorname{spt}\psi_k \subset B_k$, the theorem follows. □

Next we consider points where the gradient vanishes. We call such points *critical*. A point x_0 is called a *non-degenerate critical pont* of φ if $\nabla\varphi(x_0) = 0$ and the *Hessian determinant*

$$h_\varphi(x_0) := det(\partial_j\partial_k\varphi(x_0)) \neq 0.$$

The corresponding *Hessian matrix* is denoted by

$$H_\varphi(x_0) := (\partial_j\partial_k\varphi(x_0)).$$

Theorem 3.5. If all critical points of φ in $\operatorname{spt}\psi$ are non-degenerate, then

$$|I(\lambda)| \lesssim \lambda^{-n/2}. \tag{3.4}$$

Proof. We may assume that $\operatorname{spt}\psi \subset B(0,1)$, $\|\psi\|_\infty \leq 1$, $\|\nabla\psi\|_\infty \leq 1$ and $\|H_\varphi\|_\infty \leq 1$. We consider first the case where φ is a special quadratic polynomial, $\varphi = Q$:

$$Q(x) = x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2.$$

We use induction on n to prove that for any special quadratic polynomial Q in $\mathbb{R}^n$ as above and for any smooth ψ with $\operatorname{spt}\psi \subset B(0,1)$, $\|\psi\|_\infty \le 1$ and $\|\nabla\psi\|_\infty \le 1$,

$$|\int e^{i\lambda Q(x)}\psi(x)dx| \le C_n\lambda^{-n/2},$$

where C_n depends only on n.

The case $n=1$ follows from Corollary 3.3. Suppose the result holds for $n-1$. By Fubini's theorem

$$I(\lambda) = \lambda^{-1/2}\int e^{i\lambda(x_2^2+\dots+x_k^2-x_{k+1}^2\dots-x_n^2)}\psi_\lambda(x_2,\dots,x_n)d(x_2,\dots,x_n),$$

where

$$\psi_\lambda(x_2,\dots,x_n) = \lambda^{1/2}\int e^{i\lambda x_1^2}\psi(x_1,\dots,x_n)dx_1.$$

Corollary 3.3 tells us that $|\psi_\lambda(x_2,\dots,x_n)| \lesssim 1$ and $|\nabla\psi_\lambda(x_2,\dots,x_n)| \lesssim 1$, the latter applying Corollary 3.3 to the first order partial derivatives of ψ in place of ψ and using our boundedness assumption on the first and second order partial derivatives of ψ. The induction hypothesis gives that

$$|\int e^{i\lambda(x_2^2+\dots+x_k^2-x_{k+1}^2-\dots-x_n^2)}\psi_\lambda(x_2,\dots,x_n)d(x_2,\dots,x_n)| \lesssim \lambda^{(1-n)/2}.$$

The theorem follows from these for such quadratic polynomials.

For the general case we use the following calculus lemma, called *Morse's lemma*:

Lemma 3.6. Let $\varphi : U \to \mathbb{R}$ be a C^∞ with $U \subset \mathbb{R}^n$ open, $x_0 \in U$, such that $\varphi(x_0) = \nabla\varphi(x_0) = 0$ and $h_\varphi(x_0) \neq 0$. Then there exists a diffeomorphism $G : V \to W$ with $V, W \subset \mathbb{R}^n$ open, $0 \in V,\ x_0 \in W$, and for some $k \in \{1,\dots,n\}$,

$$\varphi \circ G(x) = \sum_{j=1}^{k-1} x_j^2 - \sum_{j=k}^{n} x_j^2 \text{ for } x \in V.$$

Proof. We may assume $x_0 = 0$. We may also assume that the matrix $H_\varphi(0)$ is diagonal with all diagonal elements non-zero. This is achieved by first diagonalizing $H_\varphi(0)$ by an orthogonal transfomation O so that $S = O^{-1}\cdot H_\varphi(0)\cdot O$ is diagonal. By direct computation using chain rule $H_{\varphi\circ O}(0) = O^T\cdot H_\varphi(0)\cdot O$. Since the transpose O^T is O^{-1}, we have $H_{\varphi\circ O}(0) = S$, which justifies our assumption.

Under this assumption, $\partial_1\varphi(0) = 0$ and $\partial_1^2\varphi(0) \neq 0$. By implicit function theorem there is a smooth function $g : W_1 \to \mathbb{R}, W_1 \subset \mathbb{R}^{n-1}$ open, $0 \in W_1$, such that $g(0) = 0$ and

$$\partial_1\varphi(g(\tilde{x}),\tilde{x}) = 0 \text{ for } \tilde{x} = (x_2,\dots,x_n) \in W_1,$$

and $\partial_1\varphi(x_1,\tilde{x}) \neq 0$ when $(x_1,\tilde{x}) \in U, \tilde{x} \in W_1$ and $x_1 \neq g(\tilde{x})$. Let $\psi = \varphi\circ F, F(x) = (x_1 + g(\tilde{x}),\tilde{x})$. Then by chain rule $\partial_1\psi(0,\tilde{x}) = 0$ and $\partial_1^2\psi(0,\tilde{x}) \neq 0$ for $\tilde{x} \in W_1$ and by Taylor's theorem, taking W_1 sufficiently small, we can write

$$\psi(x) = \psi(0,\tilde{x}) \pm h(x)x_1^2, h(x) > 0.$$

Define $E(x) = (\frac{x_1}{\sqrt{h(x)}}, \tilde{x})$. Then

$$\psi \circ E(x) = \pm x_1^2 + \psi(0, \tilde{x})$$

and so

$$\varphi \circ F \circ E(x) = \pm x_1^2 + \psi(0, \tilde{x}).$$

Repeating this for $\psi(0, \tilde{x})$ in place $\psi(x)$ and so on, the lemma follows. □

We can now complete the proof of Theorem 3.5. Each point of $\operatorname{spt}\psi$ has a neighbourhood where either $\nabla\psi \neq 0$ or we can perform the change of variable by a diffeomorhism G provided by Morse's lemma. Covering the whole $\operatorname{spt}\psi$ with a finite number of such neighbourhoods and using a partition unity, we can write $I(\lambda) = \sum_j I_j(\lambda)$. If j corresponds to a non-critical point, $|I_j(\lambda)| \lesssim \lambda^{-n/2}$ by Theorem 3.4. For j corresponding to the non-degenerate critical points we have

$$I_j(\lambda) = \int e^{i\lambda Q_j(x)}\psi(G_j(x))J_{G_j}(x)dx,$$

where Q_j and G_j are given by Morse's lemma. For these $|I_j(\lambda)| \lesssim \lambda^{-n/2}$ by the special case considered above. □

3.3. **Surface measures.** We shall consider Fourier transforms of measures on smooth hypersurfaces of $\mathbb{R}^n$. If σ is the surface measure on such a surface S, we shall consider measures μ of the type $d\mu = \zeta d\sigma$ where ζ is a smooth function with sufficiently small compact support. Moreover, we shall assume that $\operatorname{spt}\zeta \cap S$ is a graph of a smooth function φ over its tangent plane at a point $p \in S$. Without loss of generality we assume that $p = 0$ and the tangent plane is $\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\}$. The reader can of course easily deduce various generalizations from this basic case.

So let $U \subset \mathbb{R}^{n-1}$ be bounded and open, $0 \in U$, $\varphi : U \to \mathbb{R}$ and $\zeta : \mathbb{R}^n \to \mathbb{R}$ be smooth functions, ζ with compact support, such that

$$\begin{aligned} &S = \{(x, \varphi(x)) : x \in U\}, \\ &\varphi(0) = \nabla\varphi(0) = 0, \\ &\operatorname{spt}\zeta \subset \{(x, t) : x \in U\}. \end{aligned}$$

Then the measure $\mu = \zeta\sigma$ is given by

$$\int g d\mu = \int_U g(x, \varphi(x))\psi(x)dx$$

for $g \in C_0(\mathbb{R}^n)$ where

$$\psi(x) = \zeta(x, \varphi(x))\sqrt{1 + |\nabla\varphi(x)|^2}.$$

Thus the Fourier transform of μ is, writing $\xi = (\tilde{\xi}, \xi_n)$,

$$\hat{\mu}(\xi) = \int e^{-2\pi i(\tilde{\xi}\cdot x + \xi_n\varphi(x))}\psi(x)dx.$$

In order to obtain the optimal decay $|\xi|^{(1-n)/2}$ as in the case of the sphere, we need to make curvature assumptions. The *Gaussian curvature* of S at $(x, \varphi(x))$ is the Hessian determinant $h_\varphi(x)$, which is the the product of the principal curvatures, that is, the eigenvalues of $H_\varphi(x)$.

Theorem 3.7. With the above assumptions, if $h_\varphi(x) \neq 0$ for $x \in U$, then

$$|\hat{\mu}(\xi)| \lesssim |\xi|^{(1-n)/2} \text{ for } \xi \in \mathbb{R}^n.$$

Proof. Let $\xi = \lambda\eta$ with $\lambda = |\xi| > 0$ and $|\eta| = 1$, and

$$\varphi_\eta(x) = -2\pi(\eta_1 x_1 + \dots \eta_{n-1}x_{n-1} + \eta_n\varphi(x)), x \in U.$$

Then we need to show that

$$|\hat{\mu}(\xi)| = |\int e^{i\lambda\varphi_\eta(x)}\psi(x)dx| \lesssim_\eta \lambda^{(1-n)/2}.$$

The implicit constant may a priori depend on η, since the integral is a continous function of η and hence attains a maximum on S^{n-1}.

We have

$$\nabla\varphi_\eta(x) = -2\pi((\eta_1, \dots, \eta_{n-1}) + \eta_n\nabla\varphi(x))$$

and

$$H_{\varphi_\eta}(x) = -2\pi\eta_n H_\varphi(x).$$

If $\eta_n = 0$, $\nabla\varphi_\eta(x) \neq 0$ for all $x \in U$, and the required estimate follows from Theorem 3.4. If $\eta_n \neq 0$, the assumption $h_\varphi(x) \neq 0$ for $x \in U$ implies that $h_{\varphi_\eta}(x) \neq 0$ for $x \in U$, and the required estimate follows from Theorem 3.5.

□

4. Restriction problems

The presentation of this section is mainly based on [W]. This topic is also discussed in the books [G1], [So] and [S].

4.1. **The problem.** When does $\hat{f}|S^{n-1}$ make sense? If $f \in L^1(\mathbb{R}^n)$ it obviously does, since $\hat{f}$ is a continuous function and as such defined uniquely at every point. If $f \in L^2(\mathbb{R}^n)$ it obviously doesn't, since Fourier transform is an isometry of $L^2(\mathbb{R}^n)$ onto itself and consequently $\hat{f}$ is only defined almost everywhere and nothing more can be said. In this section we shall see that for $f \in L^p(\mathbb{R}^n)$ the restriction $\hat{f}|S^{n-1}$ does make sense also for some $1 < p < 2$. This follows immediately if we have an inequality

$$\|\hat{f}\|_{L^q(S^{n-1})} \leq C_{p,q}\|f\|_p \tag{4.1}$$

valid for all $f \in \mathcal{S}$. Then by denseness of $\mathcal{S}$ in L^p $\hat{f}$ is defined as an L^q-function in S^{n-1} satisfying (4.1). That is, the linear operator $f \mapsto \hat{f}$ has a unique continuous extension to $L^p(\mathbb{R}^n) \to L^q(S^{n-1})$. The restriction problems asks for which p and q (4.1) holds. It is open in in full generality, but we shall prove a sharp result when $q = 2$.

By duality, (4.1) is equivalent with

$$\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq C_{p,q}\|f\|_{L^{q'}(S^{n-1})}. \tag{4.2}$$

Here p' and q' are conjugate exponents of p and q and $\hat{f}$ means the Fourier transform of the measure $f\sigma^{n-1}$. In the case $q = 2$ this equivalence is contained in the following more general proposition:

Proposition 4.1. Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ with compact support. The following are equivalent for any $1 \leq q \leq \infty$ and $0 < C < \infty$:

(1) $\|\widehat{f\mu}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mu)}$ for all $f \in L^2(\mu)$.
(2) $\|\hat{f}\|_{L^2(\mu)} \leq C\|f\|_{L^{q'}(\mathbb{R}^n)}$ for all $f \in \mathcal{S}$.
(3) $\|\hat{\mu} * f\|_{L^q(\mathbb{R}^n)} \leq C^2\|f\|_{L^{q'}(\mathbb{R}^n)}$ for all $f \in \mathcal{S}$.

The proof can be based on the following lemma, the proofs of the above statements are left as exercises:

Lemma 4.2. Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ with compact support. Then for all $f, g \in \mathcal{S}$,

$$\int \hat{f}\bar{\hat{g}}d\mu = \int (\hat{\mu} * \bar{g})f.$$

In order to solve the restriction problem it would be enough to prove the sharp inequality (4.1) when $p = 1$ (or (4.2) when $p' = \infty$). The rest would follow by interpolation between 2 and 1 (or 2 and ∞) using Theorem 4.4 below. The following is called *restriction conjecture*:

Conjecture 4.3. $\|\hat{f}\|_{L^q(\mathbb{R}^n)} \leq C_{p,q}\|f\|_{L^\infty(S^{n-1})}$ for $q > 2n/(n-1)$.

This would be sharp; it suffices to take $f = 1$ and verify that $\sigma^{n-1} \notin L^{2n/(n-1)}$. The latter follows from (2.8) and the asymptotic formula (2.7) for the Bessel functions.

4.2. **Tomas-Stein restriction theorem.** We shall now prove the following restriction theorem due to Tomas and Stein from the 1970's:

Theorem 4.4. We have for $f \in L^2(S^{n-1})$,

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \leq C_q\|f\|_{L^2(S^{n-1})}$$

for $q \geq (2n+2)/(n-1)$. The lower bound $(2n+2)/(n-1)$ is the best possible.

Proof. We shall prove the inequality only for $q > (2n+2)/(n-1)$. For the end point, see [S]. By Proposition 4.1 the claim is equivalent to

$$\|\widehat{\sigma^{n-1}} * f\|_q \leq C_q^2\|f\|_{q'}. \tag{4.3}$$

We have

$$\sigma^{n-1}(B(x,r)) \leq Cr^{n-1} \text{ for } x \in \mathbb{R}^n, r > 0, \tag{4.4}$$

and by (2.8),

$$|\widehat{\sigma^{n-1}}(\xi)| \leq C(1+|\xi|)^{(1-n)/2} \text{ for } \xi \in \mathbb{R}^n. \tag{4.5}$$

In fact, these are the only properties of σ^{n-1} we are going to use. Thus the result holds for the more general surfaces considered at the end of the previous section.

Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(x) = 1$, when $|x| \geq 1$, and $\chi(x) = 0$, when $|x| \leq 1/2$, and set

$$\varphi(x) = \chi(2x) - \chi(x).$$

Then

$$\operatorname{spt}\varphi \subset \{x \in \mathbb{R}^n : 1/4 \leq |x| \leq 1\},$$

and

$$\sum_{j=0}^\infty \varphi(2^{-j}x) = 1 \text{ when } |x| \geq 1.$$

Write

$$\widehat{\sigma^{n-1}} = K + \sum_{j=0}^\infty K_j,$$
$$K_j(x) = \varphi(2^{-j}x)\widehat{\sigma^{n-1}}(x),$$
$$K(x) = (1 - \sum_{j=0}^\infty \varphi(2^{-j}x))\widehat{\sigma^{n-1}}(x).$$

Then K and K_j are C^∞-functions with compact support, $\operatorname{spt} K \subset B(0,1)$ (closed ball), and $\operatorname{spt} K \subset \{x : 2^{j-2} \leq |x| \leq 2^j\}$. Young's inequality for convolution (see for example [G1]) states that

$$\|g * h\|_q \leq \|g\|_p \|h\|_r \text{ when } 1 \leq p,q,r \leq \infty, \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Applying this with $g = K, h = f, p = q/2$ and $r = q'$ and using $\|K\|_p \leq \|K\|_\infty \lesssim 1$, we obtain

(4.6) $$\|K * f\|_q \leq C\|f\|_{q'}.$$

For $j = 0, 1, \dots$, we have by (2.8),

$$\|K_j\|_\infty \leq C2^{-j(n-1)/2}.$$

Thus

$$|\|K_j * f\|_\infty \leq C2^{-j(n-1)/2}\|f\|_1.$$

Define $\psi, \psi_j \in \mathcal{S}$ by

$$\psi = \hat{\varphi}, \psi_j(x) = 2^{nj}\psi(2^jx).$$

By (1.19), as $\widehat{\sigma^{n-1}} = \mathcal{F}^{-1}(\sigma^{n-1})$, $\widehat{K_j} = \psi_j * \sigma^{n-1}$. Hence for $N = 1, 2, \dots$,

$$|\widehat{K_j}(\xi)| = |2^{nj} \int \psi(2^j(\xi - \eta))d\sigma^{n-1}\eta| \leq C_N 2^{nj} \int (1 + 2^j|\xi - \eta|)^{-N} d\sigma^{n-1}\eta,$$

since $\psi \in \mathcal{S}$. Thus

$$|\widehat{K_j}(\xi)| \leq$$

$$C_N 2^{nj}(\int_{B(\xi,2^{-j})}(1+2^j|\xi-\eta|)^{-N}d\sigma^{n-1}\eta + \sum_{k=0}^{\infty}\int_{B(\xi,2^{k+1-j})\setminus B(\xi,2^{k-j})}(1+2^j|\xi-\eta|)^{-N}d\sigma^{n-1}\eta \leq$$

$$C_N 2^{nj}(\sigma^{n-1}(B(\xi,2^{-j}))+\sum_{k=0}^{\infty}2^{-Nk}\sigma^{n-1}(B(\xi,2^{k+1-j})) \lesssim$$

$$2^{nj}2^{-j(n-1)}+\sum_{k=0}^{\infty}2^{-Nk}2^{(k-j)(n-1)})) \lesssim 2^j$$

choosing $N = n$.

Since for $f, g \in \mathcal{S}$,

$$\|g * f\|_2 = \|\widehat{g * f}\|_2 = \|\hat{g}\hat{f}\|_2 \leq \|\hat{g}\|_\infty \|\hat{f}\|_2,$$

we get

$$\|K_j * f\|_2 \lesssim 2^j \|f\|_2.$$

Above we had

$$\|K_j * f\|_\infty \leq C 2^{-j(n-1)/2}\|f\|_1.$$

Let $\theta \in (0,1)$ be defined by $\theta/2 + (1-\theta)/\infty = 1/q$, that is, $\theta = 2/q$. Then by the Riesz-Thorin interpolation theorem,

$$\|K_j * f\|_q \lesssim 2^{j\theta}2^{-j((n-1)/2)(1-\theta)}\|f\|_{q'} = 2^{j((n+1)/q-(n-1)/2)}\|f\|_{q'}.$$

Since $q > \frac{2n+2}{n-1}$, $(n+1)/q - (n-1)/2 < 0$, so

$$\sum_{k=0}^{\infty}\|K_j * f\|_q \lesssim \|f\|_{q'}.$$

By (4.6) we have also,

$$\|K * f\|_q \lesssim \|f\|_{q'}.$$

This and the representation $\widehat{\sigma^{n-1}} = K + \sum_{j=0}^{\infty} K_j$ give the required inequality (4.3).

Now we discuss the sharpness of the theorem by some examples. Let $e_n = (0,\dots,0,1) \in \mathbb{R}^n$ and set for $0 < \delta < 1$,

$$C_\delta = \{x \in S^{n-1} : 1 - x \cdot e_n \leq \delta^2\}.$$

Then C_δ is a spherical cap of radius roughly δ. Choose,

$$f = \chi_{C_\delta}.$$

Then

$$\|f\|_{L^2(S^{n-1})} = \sigma^{n-1}(C_\delta)^{1/2} \approx \delta^{(n-1)/2}. \tag{4.7}$$

We estimate the Fourier transform of f in the 'dual rectangular box' of C_δ;

$$R_\delta = \{\xi \in \mathbb{R}^n : |\xi_j| \leq c/\delta \text{ for } j = 1,\dots,n-1, |\xi_n| \leq c/\delta^2\}.$$

Here c is a small constant depending only on n and to be fixed later. For $\xi \in \mathbb{R}^n$,

$$|\hat{f}(\xi) = |\int_{C_\delta} e^{-2\pi i\xi\cdot x} d\sigma^{n-1}x|$$
$$= |\int_{C_\delta} e^{-2\pi i\xi\cdot(x-e_n)} d\sigma^{n-1}x| \geq \int_{C_\delta} \cos(2\pi\xi\cdot(x-e_n))d\sigma^{n-1}x.$$

We used only that $|e^{-2\pi\xi\cdot e_n}| = 1$ and that the absolute value of a complex number is at least its real part. Choosing $c < n/6$ one checks easily that

$$|2\pi\xi\cdot(x-e_n)| < \pi/3 \text{ for } x \in C_\delta, \xi \in R_\delta,$$

whence

$$\cos(2\pi\xi\cdot(x-e_n)) > 1/2 \text{ for } x \in C_\delta, \xi \in R_\delta,$$

and so

$$|\hat{f}(\xi)| \geq \sigma^{n-1}(C_\delta)/2 \text{for } \xi \in R_\delta.$$

Since $\mathcal{L}^n(R_\delta) = 2^n c^n \delta^{-n-1}$, we get

$$\|\hat{f}\|_q \geq (\sigma^{n-1}(C_\delta)/2)^q \mathcal{L}^n(R_\delta))^{1/q} \approx \delta^{n-1-(n+1)/q}.$$

Recalling (4.7) we see that in order to have

$$\|\hat{f}\|_q \lesssim \|f\|_{L^2(S^{n-1})} \approx \delta^{(n-1)/2}$$

we must have $\delta^{n-1-(n+1)/q} \lesssim \delta^{(n-1)/2}$ for small δ, which means $n-1-(n+1)/q \geq (n-1)/2$, that is, $q \geq (2n+2)/(n-1)$ as claimed. $\square$

The dual inequality for Theorem 4.4 is

$$\|\hat{f}\|_{L^2(S^{n-1})} \lesssim \|f\|_p,\ 1 \leq p \leq \frac{2n+2}{n+3},$$

which of course is also sharp. We shall illustrate the sharpness in the plane by a slightly different example. Then $\frac{2n+2}{n+3} = \frac{6}{5}$. For $0 < \delta < 1$, consider the annulus

$$A_\delta = \{\xi \in \mathbb{R}^2 : 1-\delta \leq |\xi| \leq 1+\delta\}.$$

Our inequality can be shown to be equivalent with

$$\int_{A_\delta} |\hat{f}(\xi)|^2 d\xi \lesssim \delta(\int_{\mathbb{R}^2} |f|^p)^{2/p}. \tag{4.8}$$

If $c > 0$ is small enough, the rectangle

$$R_\delta = \{\xi \in \mathbb{R}^2 : |\xi_1 - 1| \leq c\delta, |\xi_2| \leq c\sqrt{\delta}\}$$

is contained in the annulus A_δ. Let $g \in \mathcal{S}(\mathbb{R})$ with $\hat{g}(\xi) \geq 1$ when $|\xi| \leq c$ and define f by

$$f(x_1, x_2) = g(\delta x_1)e^{-2\pi i x_1} g(\sqrt{\delta}x_2)\delta^{3/2},$$

which means that

$$\hat{f}(\xi_1, \xi_2) = \hat{g}((\xi_1 - x_1)/\delta)\hat{g}(\xi_2/\sqrt{\delta}).$$

Thus $\hat{f}(\xi) \geq 1$ when $\xi \in R_\delta$. Then, if (4.8) holds,

$$\int_{R_\delta} |\hat{f}|^2 \leq \int_{A_\delta} |\hat{f}|^2 \lesssim \delta(\int_{\mathbb{R}^2} |f|^p)^{2/p}.$$

Plugging in the formulas for f and $\hat{f}$ and changing variables, we derive from this

$$\delta^{3/2} \lesssim \delta(\int_{-\infty}^{\infty} |g(\delta x_1)|^p dx_1 \int_{-\infty}^{\infty} |g(\sqrt{\delta} x_2)|^p dx_2 \delta^{3p/2})^{2/p}$$
$$\delta(\delta^{-1}\delta^{-1/2}\delta^{3p/2})^{2/p} = \delta^{4-3/p},$$

which yields the desired $p \leq 6/5$.

[S] and [W] contain much more information on the restriction problem. In particular they present the powerful method based on stationary phase (in the spirit of the previous section) and interpolation theorems. I shall not go into that, but we shall return to the restriction topic and its connections to geometric Kakeya problems in the next section.

4.3. **Applications to PDE's.** One of the main motivations to restriction results is their applications to partial differential equations. Here is a quick glance at that.

Consider the Schrödinger equation:

$$\frac{\partial}{\partial t} u(x,t) = i\Delta_x u(x,t),\ u(x,0) = f(x),\ (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

Its solution is given by

$$u(x,t) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi - 2\pi t|\xi|^2)} \hat{f}(\xi) d\xi.$$

Let

$$S = \{(x, 2\pi|x|^2) : x \in \mathbb{R}^n\}$$

and let σ be the surface measure on S. Defining g by

$$\hat{f}(\xi) = g(\xi, 2\pi|\xi|^2)\sqrt{1 + (4\pi|\xi|)^2},$$

where $(4\pi|\xi|)^2 = |\nabla\varphi(\xi)|^2$ with $\varphi(\xi) = 2\pi|\xi|^2$, we have

$$u(x,t) = \widehat{g\sigma}(x,t)$$

and the restriction theorems give for certain values of p,

$$\|\widehat{g\sigma}\|_{L^p(\mathbb{R}^n\times\mathbb{R})} \lesssim \|g\|_{L^2(\sigma)}.$$

But

$$\|g\|_{L^2(\sigma)} \approx \|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)},$$

so

$$\|u\|_{L^p(\mathbb{R}^n\times\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

This method with variations applies to many other equations. For the wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = \Delta_x u(x,t),\ u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = f(x),\ (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

there is a similar connection with the cone $\{(x,t) : |x| = t\}$ and one needs restriction theorems for surfaces with 0 Gaussian curvature.

5. Kakeya problems

The main reference for this section is [W], but Stein also discusses this topic in [S].

5.1. **Besicovitch sets.** We say that a compact set in $\mathbb{R}^n$ is a *Besicovitch set*, or a Kakeya set, if it has zero Lebesgue measure and it contains a line segment of unit length in every direction. It is not clear that such sets exist but they do in every $\mathbb{R}^n, n \geq 2$. Besicovitch was the first to construct such a set in 1919. In doing this he also solved a problem of Kakeya: in how small, in terms of area, plane domain can a unit segment be turned around continuously. The answer is: in arbitrarily small. Such constructions can be found in [F], [Gu], [S] and [W]. Fefferman was the first to use these constructions to problems in Fourier transform in 1971, for the ball multiplier problem. We shall return to this later. In this section we shall discuss relations between Besicovitch sets and restriction problems. The pioneering work for that is Bourgain's paper [B].

How big must Besicovitch sets be? The conjecture, usually called *Kakeya conjecture*, is

Conjecture 5.1. Every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension n.

This is true for $n = 2$, and we shall prove it, and it is open for $n \geq 3$. The relation to restriction problems is:

Restriction conjecture 4.3 implies Kakeya conjecture.

We shall prove also this.

5.2. **Kakeya maximal function.** It is natural to approach these problems via a related maximal function, which also will provide a link between the two conjectures. For $a \in \mathbb{R}^n, e \in S^{n-1}$ and $\delta > 0$, define the tube $T_e^\delta(a)$ with center a, direction e, length 1 and radius δ:

$$T_e^\delta(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot e| \leq 1/2, |x - a - ((x-a)\cdot e)e| \leq \delta\}.$$

Definition 5.2. The *Kakeya maximal function* with width δ of $f \in L^1_{loc}(\mathbb{R}^n)$ is the function

$$f_\delta^* : S^{n-1} \to [0,\infty],$$

$$f_\delta^*(e) = \sup_{a\in\mathbb{R}^n} \frac{1}{\mathcal{L}^n(T_e^\delta(a))} \int_{T_e^\delta(a)} |f| d\mathcal{L}^n.$$

We have the trivial proposition:

Proposition 5.3. For all $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\|f_\delta^*\|_\infty \leq \|f\|_\infty \text{ and } \|f_\delta^*\|_\infty \lesssim \delta^{1-n}\|f\|_1.$$

If $n \geq 2$, as we shall now always assume, and if $p < \infty$, there can be no inequality

$$\|f_\delta^*\|_{L^q(S^{n-1})} \leq C\|f\|_p \text{ for all } f \in L^p(\mathbb{R}^n)$$

with C independent of δ. This follows from the existence of Besicovitch sets: let $B \subset \mathbb{R}^n$ be such a compact set (with $\mathcal{L}^n(B) = 0$) and let

$$f = \chi_{B_\delta},\ B_\delta = \{x \in \mathbb{R}^n : dist(x, B) \leq \delta\}.$$

Then $f_\delta^*(e) = 1$ for all $e \in S^{n-1}$, so $\|f_\delta^*\|_{L^q(S^{n-1})} \approx 1$ but $\|f\|_p = \mathcal{L}^n(B_\delta)^{1/p} \to 0$ as $\delta \to 0$. Consequently we look for inequalities like:

$$\|f_\delta^*\|_{L^p(S^{n-1})} \leq C_\varepsilon \delta^{-\varepsilon}\|f\|_p \text{ for all } \varepsilon > 0, f \in L^p(\mathbb{R}^n). \tag{5.1}$$

Even this cannot hold if $p < n$: let $f = \chi_{B(0,\delta)}$. Since $B(0, \delta) \subset T_e^\delta(0)$, we have for all $e \in S^{n-1}$,

$$f_\delta^*(e) = \frac{\mathcal{L}^n(B(0,\delta))}{\mathcal{L}^n(T_e^\delta(0))} \gtrsim \delta.$$

But

$$\|f\|_p = \mathcal{L}^n(B(0, \delta))^{1/p} \approx \delta^{n/p},$$

and $\delta >> \delta^{n/p}$ for small δ if $p < n$. Kakeya maximal conjecture wishes for the next best thing:

Conjecture 5.4. (5.1) holds if $p = n$, that is,

$$\|f_\delta^*\|_{L^n(S^{n-1})} \leq C_\varepsilon \delta^{-\varepsilon}\|f\|_n \text{ for all } \varepsilon > 0, f \in L^n(\mathbb{R}^n).$$

We shall see that this holds in $\mathbb{R}^2$ even with a logarithmic factor in place of $C_\varepsilon \delta^{-\varepsilon}$. In $\mathbb{R}^n, n \geq 3$, the question is open. We shall first prove a connection to the dimension of Besicovitch sets.

Theorem 5.5. If (5.1) holds for some $p, 1 \leq p < \infty$, then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is n. In particular, Conjecture 5.4 implies Kakeya conjecture.

Proof. Let $B \subset \mathbb{R}^n$ be a Besicovitch set. Let $0 < \alpha < n$ and let $B_j = B(x_j, r_j), j = 1, 2, \ldots,$ be balls such that $r_j \leq 1/100$ and $B \subset \cup_j B_j$. It suffices to show that $\sum_j r_j^\alpha \gtrsim 1$.

For $e \in S^{n-1}$ let $I_e \subset B$ be a unit segment parallel to e. For $k = 1, 2, \ldots,$ set

$$J_k = \{j : 2^{-k} \leq r_j < 2^{1-k}\},$$

and

$$S_k = \{e \in S^{n-1} : \mathcal{H}^1(I_e \cap \cup_{j\in J_k} B_j) \geq \frac{1}{100k^2}\}.$$

Here $\mathcal{H}^1$ is the one-dimensional Hausdorff measure (length measure). Since $\sum_k \frac{1}{100k^2} < 1$ and

$$\sum_k \mathcal{H}^1(I_e \cap \cup_{j\in J_k} B_j) \geq \mathcal{H}^1(I_e) \geq 1,$$

we have

$$\bigcup_k S_k = S^{n-1};$$

if there were some $e \in S^{n-1} \setminus \bigcup_k S_k$, we would have $\mathcal{H}^1(I_e \cap \cup_{j\in J_k} B_j) < \frac{1}{100k^2}$ for all k, and then

$$\sum_k \mathcal{H}^1(I_e \cap \cup_{j\in J_k} B_j) < \sum_k \frac{1}{100k^2} < 1,$$

which is impossible.

Let

$$f = \chi_{F_k} \text{ with } F_k = \cup_{j\in J_k} B(x_j, 10r_j).$$

If $e \in S_k$, then, letting a_e be the mid-point of I_e,

$$\mathcal{L}^n(T_e^{2^{-k}}(a_e) \cap F_k) \geq \frac{1}{100k^2}\mathcal{L}^n(T_e^{2^{-k}}(a_e)),$$

whence

$$\|f^*_{2^{-k}}\|_p \gtrsim \frac{1}{k^2}\sigma^{n-1}(S_k)^{1/p}.$$

The assumption (5.1) gives

$$\|f^*_{2^{-k}}\|_p \leq C_\varepsilon 2^{k\varepsilon}\|f\|_p \leq C_\varepsilon 2^{k\varepsilon}(\# J_k 2^{(1-k)n})^{1/p}.$$

Combining these two inequalities,

$$\sigma^{n-1}(S_k) \lesssim_\varepsilon 2^{k\varepsilon p}k^{2p}2^{-kn}\# J_k \lesssim_\varepsilon 2^{-k(n-2\varepsilon p)}\# J_k,$$

and finally

$$\sum_j r_j^{n-2\varepsilon p} \gtrsim \sum_k \# J_k 2^{-k(n-2\varepsilon p)} \gtrsim \sum_k \sigma^{n-1}(S_k) \gtrsim 1.$$

Choosing ε such that $n - 2p\varepsilon > \alpha$ we get

$$\sum_j r_j^\alpha \gtrsim 1$$

as required.

□

Remark 5.6. An obvious modification of the above proof gives that if

$$\|f^*_\delta\|_{L^p(S^{n-1})} \leq C_\beta \delta^{-\beta}\|f\|_p \text{ for all} f \in L^p(\mathbb{R}^n),$$

then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $n - p\beta$.

We shall now prove a fairly sharp estimate in the plane:

Theorem 5.7. For all $\delta > 0$ and $f \in L^2(\mathbb{R}^2)$,

$$\|f^*_\delta\|_{L^2(S^1)} \leq C\sqrt{\log(1/\delta)}\|f\|_{L^2(\mathbb{R}^2)},$$

with C independent of δ and f.

Proof. We may assume that f is non-negative and has compact support. Changing variable and using the symmetricity of $T_e^\delta(0)$ we have

$$f_\delta^*(e) = \sup_{a\in\mathbb{R}^2} \frac{1}{\mathcal{L}^2(T_e^\delta(a))} \int_{T_e^\delta(a)} f d\mathcal{L}^2$$
$$= \sup_{a\in\mathbb{R}^2} \frac{1}{2\delta} \int_{T_e^\delta(0)} f(a-x) d\mathcal{L}^2 x = \sup_{a\in\mathbb{R}^2} \varrho_\delta^e * f(a),$$

where

$$\varrho_\delta^e = \frac{1}{2\delta}\chi_{T_e^\delta(0)}.$$

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\operatorname{spt}\hat{\varphi} \subset [-1,1],\ \varphi \geq 0$ and $\varphi(x) \geq 1$ when $|x| \leq 1$ (check that such a φ exists as an exercise if you havn't met before). Define

$$\psi(x) = \varphi(x_1)\frac{1}{\delta}\varphi(x_2/\delta), x = (x_1,x_2) \in \mathbb{R}^2.$$

Then $\hat{\psi}(\xi_1,\xi_2) = \hat{\varphi}(\xi_1)\hat{\varphi}(\delta\xi_2)$ and so

$$\operatorname{spt}\hat{\psi} \subset [-1,1]\times[-1/\delta,1/\delta]. \tag{5.2}$$

Since $\varphi(x_1) \geq 1$ and $\varphi(x_2/\delta) \geq 1$ when $|x_1| \leq 1$ and $|x_2| \leq \delta$, we have $\varrho_\delta^{e_1} \leq \psi$, with $e_1 = (1,0)$, and so

$$f_\delta^*(e_1) \leq \sup_{a\in\mathbb{R}^2} \psi * f(a). \tag{5.3}$$

For $e \in S^1$, let $g_e : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation, $g_e(x) = (e\cdot x, e^\perp \cdot x)$, where $e\cdot e^\perp = 0$. Then $g_e(x) \in T_{e_1}^\delta(0)$ if and only if $x \in T_e^\delta(0)$. Hence defining $\psi_e = \psi\circ g_e$, we get from (5.3)

$$f_\delta^*(e) \leq \sup_{a\in\mathbb{R}^2} \psi_e * f(a).$$

Using the fast decay of $\psi_e * f$ and Schwartz's inequality, this leads to

$$f_\delta^*(e) \leq \|\psi_e * f\|_\infty \leq \|\widehat{\psi_e * f}\|_1 = \int |\widehat{\psi_e}||\hat{f}|$$
$$\leq (\int |\widehat{\psi_e}(\xi)||\hat{f}(\xi)|^2(1+|\xi|)d\xi)^{1/2}(\int \frac{|\widehat{\psi_e}(\xi)|}{1+|\xi|}d\xi)^{1/2}.$$

Since $\widehat{\psi_e}(\xi) = \widehat{\psi\circ g_e}(\xi) = \hat{\psi}(g_e(\xi))$, we get from (5.2)

$$\operatorname{spt}\widehat{\psi_e} \subset R_e := g_e^{-1}([-1,1]\times[-1/\delta,1/\delta]),$$

and so

$$\int \frac{|\widehat{\psi_e}(\xi)|}{1+|\xi|}d\xi \lesssim \int_{R_e} \frac{1}{1+|\xi|}d\xi \approx \int_{-1/\delta}^{1/\delta} \frac{1}{1+t}dt \approx \log(\frac{1}{\delta}).$$

Thus

$$\|f_\delta^*\|_{L(S^1)}^2 \lesssim \log(\frac{1}{\delta}) \int_{S^1}\int_{\mathbb{R}^2} |\widehat{\psi_e}(\xi)||\hat{f}(\xi)|^2(1+|\xi|)d\xi d\sigma^1 e$$
$$= \log(\frac{1}{\delta}) \int_{\mathbb{R}^2}(\int_{S^1} |\widehat{\psi_e}(\xi)|d\sigma^1 e)|\hat{f}(\xi)|^2(1+|\xi|)d\xi.$$

Using again that $\operatorname{spt}\widehat{\psi_e} \subset R_e$ we get for all $\xi \in \mathbb{R}^2$,

$$\sigma^1(\{e \in S^1 : \widehat{\psi_e}(\xi) \neq 0\} \leq \sigma^1(\{e \in S^1 : \xi \in R_e\} \lesssim \frac{1}{1+|\xi|}.$$

The last inequality is a simple geometric fact. Consequently,

$$\int |\widehat{\psi_e}(\xi)| d\sigma^1 e \lesssim \frac{1}{1+|\xi|}$$

and

$$\|f_\delta^*\|_{L(S^1)}^2 \lesssim \log(\frac{1}{\delta}) \int_{\mathbb{R}^2} |\frac{1}{1+|\xi|} |\hat{f}(\xi)|^2 (1+|\xi|) d\xi = \log(\frac{1}{\delta}) \|f\|_2^2.$$

□

Remark 5.8. In $\mathbb{R}^n, n \geq 3$, a modification of the above proof gives that if

$$\|f_\delta^*\|_{L^2(S^{n-1})} \lesssim \delta^{(2-n)/2} \|f\|_2,$$

where the exponent $(2-n)/2$ is the best possible.

The above proof is due to Bourgain from 1991. Originally this (and many other Kakeya-type things) was proved by Córdoba in 1977 by a geometric method without using Fourier transform. See [W] for Córdoba's proof.

Combining Theorems 5.5 and 5.7 we obtain

Corollary 5.9. All Besicovitch sets in $\mathbb{R}^2$ have Hausdorff dimension 2.

This corollary was first proved by Davies in 1971 using Marstrand's projection and line intersection theorems for Hausdorff dimension.

In the following lemma gives a discretized version of Kakeya type inequalities. This lemma is also an essential ingredient in the above mentioned Córdoba's proof of Theorem 5.7.

Lemma 5.10. Let $1 < p < \infty, p' = \frac{p}{p-1}, 0 < \delta < 1$ and $0 < \Lambda < \infty$. Suppose that

$$\|\sum_{k=1}^{m} t_k \chi_{T_k}\|_{p'} \leq \Lambda$$

whenever $\{e_1, \dots, e_m\} \subset S^{n-1}$ is a maximal δ-separated subset of S^{n-1}, $t_1, \dots, t_m$ are positive numebrs with

$$\delta^{n-1} \sum_{k=1}^{m} t_k^{p'} \leq 1,$$

$a_1, \dots, a_m \in \mathbb{R}^n$ and $T_k = T_{e_k}^\delta(a_k)$. Then there is positive number C depending only on n such that

$$\|f_\delta^*\|_{L^p(S^{n-1})} \leq C\Lambda \|f\|_p \text{ for all } f \in L^p(\mathbb{R}^n).$$

By $\{e_1, \dots, e_m\} \subset S^{n-1}$ being a maximal δ-separated subset of S^{n-1} we mean of course that $|e_j - e_k| \geq \delta$ for $j \neq k$ and for every $e \in S^{n-1}$ there is some k for which $|e - e_k| < \delta$.

Proof. Let $\{e_1,\dots,e_m\}\subset S^{n-1}$ be a maximal δ-separated subset of S^{n-1}. If $e\in S^{n-1}\cap B(e_k,\delta)$, then $f^*_\delta(e)\le Cf^*_\delta(e_k)$ with C depending only on n, because any $T^\delta_e(a)$ can be covered with boundedly many tubes $T^\delta_{e_k}(a_j)$. Hence

$$\begin{aligned}\|f^*_\delta\|^p_{L^p(S^{n-1})}&\le\sum_k\int_{B(e_k,\delta)}f^{*p}_\delta d\sigma^{n-1}\\&\le\sum_k C^pf^*_\delta(e_k)^p\sigma^{n-1}(B(e_k,\delta))\lesssim\sum_k f^*_\delta(e_k)^p\delta^{n-1}.\end{aligned}$$

By the duality of l^p and $l^{p'}$, for any $a_j\ge0, j=1,\dots,m$,

$$(\sum_{j=1}^m a_j^p)^{1/p}=\max\{\sum_{j=1}^m a_jb_j : b_j\ge0,\sum_{j=1}^m b_j^{p'}=1\}.$$

Applying this to $a_k=\delta^{(n-1)/p}f^*_\delta(e_k)$ we get

$$\begin{aligned}\|f^*_\delta\|_{L^p(S^{n-1})}&\lesssim(\sum_k(\delta^{(n-1)/p}f^*_\delta(e_k))^p)^{1/p}\\&=\sum_k\delta^{(n-1)/p}f^*_\delta(e_k)b_k=\delta^{(n-1)}\sum_k t_kf^*_\delta(e_k))\end{aligned}$$

where $\sum_k b_k^{p'}=1, t_k=\delta^{(1-n)/p'}b_k$, and so $\delta^{n-1}\sum_k t_k^{p'}=1$. Therefore

$$\|f^*_\delta\|_{L^p(S^{n-1})}\lesssim\delta^{(n-1)}\sum_k t_k\frac{1}{\mathcal{L}^n(T^\delta_{e_k}(a_k))}\int_{T^\delta_{e_k}(a_k)}|f|d\mathcal{L}^n$$

for some $a_k\in\mathbb{R}^n$. Since $\mathcal{L}^n(T^\delta_{e_k}(a_k))\approx\delta^{n-1}$, we obtain by Hölder's inequality

$$\begin{aligned}\|f^*_\delta\|_{L^p(S^{n-1})}&\lesssim\sum_k t_k\int_{T^\delta_{e_k}(a_k)}|f|d\mathcal{L}^n=\int(\sum_k t_k\chi_{T^\delta_{e_k}(a_k)})|f|d\mathcal{L}^n\\&\|\sum_k t_k\chi_{T^\delta_{e_k}(a_k)}\|_{p'}\|f\|_p\le\Lambda\|f\|_p.\end{aligned}$$

□

5.3. **Restriction implies Kakeya.** Next we prove that the restriction inequality

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)}\le C_q\|f\|_{L^q(S^{n-1})}\text{ for }q>2n/(n-1). \tag{5.4}$$

implies the Kakeya maximal conjecture 5.4. It can be shown, see [B], that Restriction conjecture 4.3 is equivalent with (5.4) (that is, $\|f\|_{L^q(S^{n-1})}$ can be replaced by $\|f\|_\infty$ on the right hand side), so by Theorem 5.5 Restriction conjecture implies Kakeya conjecture.

For the proof we need a probabilistic result called Khintchin's inequality. It has also many other applications in analysis, for example, one can prove the sharpness of the Hausdorff-Young inequality using it. Let $\omega_j, j=1,2,\dots$, be independent random variables on a probability space (Ω,P) taking values ±1 with equal probability $1/2$. One can take for example $\Omega=\{-1,1\}^{\mathbb{N}},\omega_j((x_k))=x_j$, and P the natural measure on Ω, the infinite product of the measures $\frac{1}{2}(\delta_{-1}+\delta_1)$. Denote by

$\mathbb{E}(f)$ the expectation (P-integral) of the random variable f. The indepence of the ω_j's implies that

$$\mathbb{E}(\omega_j\omega_k) = \mathbb{E}(\omega_j)\mathbb{E}(\omega_k) = 0 \text{ for } j \neq k,$$

and that for any Borel function $g : \mathbb{R} \to \mathbb{R}$ and any finite subset J of $\mathbb{N}$, the random variables $g \circ \omega_j, j \in J$, are independent and

$$\mathbb{E}(\Pi_{j\in J} g \circ \omega_j) = \Pi_{j\in J}\mathbb{E}(g \circ \omega_j).$$

Theorem 5.11. For any $a_1, \dots, a_N \in \mathbb{C}$ and $0 < p < \infty$,

$$\mathbb{E}(|\sum_{j=1}^{N} \omega_j a_j|^p) \approx_p (|\sum_{j=1}^{N} |a_j|^2)^{p/2}.$$

Proof. We shall only prove this $1 < p < \infty$, which is the only case we shall need. If $p = 2$, the claim follows from independence as equality. Next we prove the inequality "$\lesssim$". We may obviously assume that the a_j's are real. Let $t > 0$. Then by the independence

$$\mathbb{E}(e^{t\sum_j a_j\omega_j}) = \Pi_j\mathbb{E}(e^{ta_j\omega_j}) = \Pi_j\mathbb{E}(\frac{1}{2}(e^{ta_j\omega_j} + e^{-ta_j\omega_j}).$$

The elementary inequality $\frac{1}{2}(e^x + e^{-x}) \leq e^{x^2/2}$ implies that

$$\mathbb{E}(e^{t\sum_j a_j\omega_j}) \leq e^{(t^2/2)\sum_j a_j^2}.$$

This gives for all $t > 0, \lambda > 0$, by Chebychev's inequality

$$\begin{aligned} P(\sum_j a_j\omega_j \geq \lambda) &= P(e^{t\sum_j a_j\omega_j} \geq e^{\lambda t}) \\ &\leq e^{-\lambda t}\mathbb{E}(e^{t\sum_j a_j\omega_j}) \leq e^{-\lambda t + (t^2/2)\sum_j a_j^2}. \end{aligned}$$

Take $t = \frac{\lambda}{\sum_j a_j^2}$. Then

$$P(\sum_j a_j\omega_j \geq \lambda) \leq e^{-\frac{\lambda^2}{2\sum_j a_j^2}}$$

and so

$$P(|\sum_j a_j\omega_j| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2\sum_j a_j^2}}.$$

Applying this and the formula (which follows from Fubini's theorem)

$$\mathbb{E}(|f|^p) = p\int_0^\infty \lambda^{p-1}e^{-\frac{\lambda^2}{2\sum_j a_j^2}} P(|f| \geq \lambda)d\lambda,$$

we get by a change of variable

$$\mathbb{E}(|\sum_j a_j\omega_j|^p) \leq 2p\int_0^\infty \lambda^{p-1}e^{-\frac{\lambda^2}{2\sum_j a_j^2}} d\lambda = c(p)(\sum_j a_j^2)^{p/2},$$

which is the desired inequality.

To prove the opposite inequality we use duality. Let $q = \frac{p}{p-1}$. Then by the two previous cases, $p = 2$ and "$\lesssim$", and by Hölder's inequality,

$$\sum_j |a_j|^2 = \mathbb{E}(|\sum_j a_j\omega_j|^2) \leq \mathbb{E}(|\sum_j a_j\omega_j|^p)^{1/p}\mathbb{E}(|\sum_j a_j\omega_j|^q)^{1/q}$$
$$\lesssim (\sum_j |a_j|^2)^{1/2}\mathbb{E}(|\sum_j a_j\omega_j|^p)^{1/p}$$

which yields

$$\mathbb{E}(|\sum_j a_j\omega_j|^p)^{1/p} \gtrsim (\sum_j |a_j|^2)^{1/2}$$

and proves the theorem. □

Theorem 5.12. (5.4) implies Conjecture 5.4

Proof. Let $\{e_1, \ldots, e_m\} \subset S^{n-1}$ be a maximal δ-separated set. Then $m \approx \delta^{1-n}$. Let $\varepsilon > 0$ and $1 < p < n$ such that with $p' = \frac{p}{p-1}$, $0 < 2(n-1-n/p') < \varepsilon$. This can be done since $n/p' \to n-1$ as $p \to n$. Let $a_1, \ldots, a_m \in \mathbb{R}^n$ and $t_1, \ldots, t_m > 0$ with

$$\delta^{n-1}\sum_{k=1}^m t_k^{p'} \leq 1.$$

Let $T_k = T^\delta_{e_k}(a_k)$. We shall show that

$$\|\sum_k t_k\chi_{T_k}\|_{p'} \leq C_1(n)\delta^{-\varepsilon}. \tag{5.5}$$

By Lemma 5.10 this implies (with $C = C(n)$)

$$\|f^*_\delta\|_{L^p(S^{n-1})} \leq C\delta^{-\varepsilon}\|f\|_p \text{ for all } f \in L^p(\mathbb{R}^n).$$

Interpolating this with the trivial inequlity

$$\|f^*_\delta\|_\infty \leq \|f\|_\infty$$

we shall get

$$\|f^*_\delta\|_{L^n(S^{n-1})} \leq C_\varepsilon\delta^{-\varepsilon}\|f\|_n \text{ for all } f \in L^p(\mathbb{R}^n)$$

as required. So we need to prove (5.5).

Let τ_k be the δ^{-2} dilation of T_k: τ_k is the cylinder with center $\delta^{-2}a_k$, direction e_k, length δ^{-2} and cross-section radius δ^{-1}. Let

$$S_k = \{e \in S^{n-1} : 1 - e\cdot e_k \leq C^{-2}\delta^2\}.$$

Then S_k is a spherical cap of radius $\approx C^{-1}\delta\}$ and center e_k. Here C is chosen big enough to guarantee that the S_k's are disjoint. Define f_k by

$$f_k = e^{2\pi i\delta^{-2}a_k\cdot x}\chi_{S_k}(x).$$

Then $\|f_k\|_\infty = 1, \operatorname{spt} f_k \subset S_k$ and by the estimates we did with the example at the end of the proof of Theorem 4.4 we see that, provided C is sufficiently large, but still depending only on n,

$$\hat{f}_k(\xi) \geq \delta^{n-1} \text{ for } \xi \in \tau_k.$$

Fix $s_k \geq 0, k = 1, \dots, m$, and for $\omega = (\omega_1, \dots, \omega_m) \in \{-1, 1\}^m$ let

$$f(\omega) = \sum_{k=1}^{m} \omega_k s_k f_k.$$

We shall consider ω_k's as independent random variables taking values 1 and -1 with equal probablility, and we shall use Khintchin's inequality.

Let $1 < q < \infty$. Since the functions f_k have disjoint supports,

$$\|f(\omega)\|_{L^q(S^{n-1})}^q = \sum_{k=1}^{m} \|s_k f_k\|_{L^q(S^{n-1})}^q \approx \sum_{k=1}^{m} s_k^q \delta^{n-1}.$$

By Fubini's theorem and Khintchin's inequality,

$$\mathbb{E}(\|\widehat{f(\omega)}\|_q^q) = \int \mathbb{E}(|\widehat{f(\omega)}(\xi)|^q) d\xi$$
$$\approx \int (\sum_{k=1}^{m} s_k^2 |\widehat{f}_k(\xi)|^2)^{q/2} d\xi \gtrsim \delta^{q(n-1)} \int (\sum_{k=1}^{m} s_k^2 \chi_{\tau_k}(\xi))^{q/2} d\xi,$$

since $|\widehat{f_k}| \gtrsim \delta^{n-1} \chi_{\tau_k}$.

Let $q > \frac{2n}{n-1}$. Then by our assumption the restriction property (5.4) holds and we get

$$\|\widehat{f(\omega)}\|_q \lesssim \|f(\omega)\|_{L^q(S^{n-1})}.$$

Combining these three inequalities, we obtain

$$\delta^{q(n-1)} \int (\sum_{k=1}^{m} s_k^2 \chi_{\tau_k})^{q/2} \lesssim \delta^{(n-1)} \sum_{k=1}^{m} s_k^q.$$

We had p and t_k given in the beginning of the proof and we choose $s_k = \sqrt{t_k}$ and $q = 2p'$. Then $q > \frac{2n}{n-1}$ as $p < n$, and $\delta^{(n-1)} \sum_{k=1}^{m} s_k^q = \delta^{(n-1)} \sum_{k=1}^{m} t_k^{p'} \leq 1$. Thus

$$\delta^{2p'(n-1)} \int (\sum_{k=1}^{m} t_k \chi_{\tau_k})^{p'} \lesssim 1.$$

Changing variable $y = \delta^2 x$, τ_k goes to T_k and so

$$\delta^{2p'(n-1)} \delta^{-2n} \int (\sum_{k=1}^{m} t_k \chi_{T_k})^{p'} \lesssim 1,$$

that is,

$$\|\sum_{k=1}^{m} t_k \chi_{T_k}\|_{p'} \lesssim \delta^{2(\frac{n}{p'} - (n-1))}.$$

As $2(\frac{n}{p'} - (n-1)) > -\varepsilon$, (5.5) follows. $\square$

Corollary 5.13. (5.4) implies Kakeya conjecture (that all Besicovitch sets in $\mathbb{R}^n$ have Hausdorff dimension n).

6. STATIONARY PHASE AND RESTRICTION

This section is based on [S] and [W]. Sogge also has a lot on this topic in [So]. The presentation will be sketchy, we shall omit many details.

6.1. **Stationary phase and L^2-estimates.** Recall that in Section 3 we investigated the decay as $\lambda \to \infty$ of the integrals

$$I(\lambda) = \int e^{i\lambda\varphi(x)}\psi(x)dx,\ \lambda > 0.$$

We found that they decay as $\lambda^{-n/2}$ provided that the critical points of φ are non-degenerate on the support of ψ. In this section we allow φ and ψ to depend also on x, we denote them now by Φ and Ψ, and we look for $L^p - L^q$ estimates for the operators

$$T_\lambda f(\xi) = \int e^{i\lambda\Phi(x,\xi)}\Psi(x,\xi)f(x)dx,\ \lambda > 0. \tag{6.1}$$

As in the case of surface measures, this leads to restriction theorems via local parametrizations of the surfaces, but this time it will be fairly complicated. We shall also see how this method can be used to prove the sharp Carleson-Sjölin restriction theorem in plane.

Under the non-degeneracy of the Hessian we have a fairly simple L^2-result:

Theorem 6.1. Suppose that $\Phi : \mathbb{R}^{2n} \to \mathbb{R}$ and $\Psi : \mathbb{R}^{2n} \to \mathbb{C}$ are C^∞-functions, Ψ with compact support. If

$$\det\left(\frac{\partial^2\Phi(x,\xi)}{\partial x_j \partial \xi_k}\right) \neq 0 \text{ for } (x,\xi) \in \operatorname{spt}\Psi, \tag{6.2}$$

then the operators T_λ, satisfy

$$\|T_\lambda f\|_2 \lesssim \lambda^{-n/2}\|f\|_2 \text{ for all } f \in L^2(\mathbb{R}^n), \lambda > 0. \tag{6.3}$$

Proof. We can write

$$\|T_\lambda f\|_2^2 = \int\int K_\lambda(\xi,\zeta)f(\xi)\overline{f(\zeta)}d\xi d\zeta,$$

where

$$K_\lambda(\xi,\zeta)) = \int e^{i\lambda(\Phi(x,\xi)-\Phi(x,\zeta))}\Psi(x,\xi)\overline{\Psi(x,\zeta)}dx.$$

By Taylor's formula

$$\nabla_x(\Phi(x,\xi) - \Phi(x,\zeta)) = \left(\frac{\partial^2\Phi(x,\xi)}{\partial x_j \partial \xi_k}\right)(\xi - \zeta) + O(|\xi-\zeta|^2).$$

Assuming that the $\operatorname{spt}\Psi$ is sufficiently small we have then that for some $c > 0$,

$$|\nabla_x(\Phi(x,\xi) - \Phi(x,\zeta))| \geq c|\xi - \zeta|$$

when $(x,\xi),(x,\zeta) \in \operatorname{spt}\Psi$. We can reduce to small support for Ψ as before (in Section 3) with finite coverings. Reducing the support of Ψ further if needed we can assume that for some $j = 1,\dots,n$,

$$|\frac{\partial}{\partial x_j}(\Phi(x,\xi)-\Phi(x,\zeta))| \geq c|\xi-\zeta|$$

when $(x,\xi),(x,\zeta) \in \operatorname{spt}\Psi$. Then similar partial integrations as in Section 3 yield

$$|K_\lambda(\xi,\zeta))| \leq C_N(1+\lambda|\xi-\zeta|)^{-N},\ N=1,2,\dots,$$

for $\xi,\zeta \in \mathbb{R}^n$. Applying this with $N = n+1$ we find that

$$\int |K_\lambda(\xi,\zeta))|d\zeta \lesssim \lambda^{-n} \text{ for } \xi \in \mathbb{R}^n,$$

$$\int |K_\lambda(\xi,\zeta))|d\xi \lesssim \lambda^{-n} \text{ for } \zeta \in \mathbb{R}^n.$$

Defining

$$T_{K_\lambda}f(\zeta) = \int K_\lambda(\xi,\zeta)f(\xi)d\xi$$

we obtain form the previous inequalities and Schur's test, which we discuss below, that

$$\|T_\lambda f\|_2^2 = \int (T_{K_\lambda}f)\bar{f} \leq \|T_{K_\lambda}f\|_2\|f\|_2 \lesssim \lambda^{-n}\|f\|_2^2,$$

as required. □

Schur's test is the following general and very useful boundedness criterion:

Theorem 6.2. Let (X,μ) and (Y,ν) be measure spaces and $K: X\times Y \to \mathbb{C}$ a $\mu\times\nu$ measurable function such that

$$\int |K(x,y))|d\mu x \leq A \text{ for } y \in Y$$

and

$$\int |K(x,y))|d\nu y \leq B \text{ for } x \in X.$$

Define

$$T_K f(x) = \int K(x,y)f(y)d\nu y,\ f \in L^2(\nu).$$

Then

$$\|T_K f\|_{L^2(\mu)} \leq \sqrt{AB}\|f\|_{L^2(\nu)} \text{ for } f \in L^2(\nu). \tag{6.4}$$

Proof. This, and also the corresponding L^p-inequality, follows from Riesz-Thorin interpolation theorem, but here is an easy direct proof: We have the elementary fact

$$\sqrt{ab} = \min\{\frac{1}{2}(\varepsilon a + b/\varepsilon) : 0<\varepsilon<\infty\},\ a,b>0.$$

(6.4) follows if we can prove that

$$\int\int |K(x,y)g(x)f(y)|d\mu x d\nu y \leq \sqrt{AB}$$

whenever $\|g\|_{L^2(\mu)} = 1$ and $\|f\|_{L^2(\nu)} = 1$. To verify this we use Schwartz's inequality and the above formula for $\sqrt{ab}$:

$$\begin{aligned}
&\int\int |K(x,y)g(x)f(y)|d\mu x d\nu y \\
&\leq (\int\int |K(x,y)||f(y)|^2 d\mu x d\nu y \int\int |K(x,y)||g(x)|^2 d\mu x d\nu y)^{1/2} \\
&= \min_{\varepsilon>0} \frac{1}{2}(\varepsilon \int\int |K(x,y)||f(y)|^2 d\mu x d\nu y + \frac{1}{\varepsilon}\int\int |K(x,y)||g(x)|^2 d\mu x d\nu y) \\
&\leq \min_{\varepsilon>0} \frac{1}{2}(\varepsilon A \int |f|^2 d\nu + \frac{1}{\varepsilon} B \int |g|^2 d\mu) = \sqrt{AB}.
\end{aligned}$$

□

6.2. **From stationary phase to restriction.** Let us see now how stationary phase can be applied to restriction problems. We are interested in inequalities

$$\|\hat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \text{ for } f \in L^p(\mathbb{R}^n). \tag{6.5}$$

Here S is a smooth surface in $\mathbb{R}^n$. Assuming that S is parametrized with a smooth compactly supported function φ (6.5) reduces again to inequalities like

$$|\int |\hat{f}(x,\varphi(x))|^q \psi(x) dx|^{1/q} \lesssim |\|f\|_{L^p(\mathbb{R}^n)}, \tag{6.6}$$

where φ and ψ are compactly supported C^∞ functions $\mathbb{R}^{n-1}$ with $\psi \geq 0, \varphi(0) = 0, \nabla\varphi(0) = 0$ and $h_\varphi(0) \neq 0$. The Fourier transform of f is given by

$$\hat{f}(x,\varphi(x)) = \int_{\mathbb{R}^n} e^{-2\pi i(x\cdot\tilde{\xi}+\varphi(x)\xi_n)} f(\xi)d\xi, \tilde{\xi} = (\xi_1,\ldots,\xi_{n-1}).$$

Let η be a non-negative compactly supported C^∞ function on $\mathbb{R}^n$ with $\eta(0) = 1$ and define

$$T_\lambda f(x) = \int e^{i\lambda\Phi(x,\xi)}\Psi(x,\xi)f(\xi)d\xi,\ \lambda > 0, x \in \mathbb{R}^{n-1}, \tag{6.7}$$

where

$$\Phi(x,\xi) = -2\pi(x\cdot\tilde{\xi} + \varphi(x)\xi_n),$$
$$\Psi(x,\xi) = \psi(x)\eta(\xi).$$

Suppose we could prove

$$\|T_\lambda f\|_{L^q(\mathbb{R}^{n-1})} \lesssim \lambda^{-n/p'}\|f\|_{L^p(\mathbb{R}^n)} \tag{6.8}$$

Applying this to $f_\lambda, f_\lambda(\xi) = f(\lambda\xi)$, we get

$$(\int |\int e^{i\lambda\Phi(x,\xi)}\eta(\xi)f(\lambda\xi)d\xi|^q \psi(x)^q dx)^{1/q} \lesssim \lambda^{-n/p'}\|f_\lambda\|_p = \lambda^{-n/p-n/p'}\|f\|_p = \lambda^{-n}\|f\|_p.$$

Change of variable $\zeta = \lambda\xi$ gives, since $\lambda\Phi(x,\xi) = \Phi(x,\lambda\xi)$,

$$(\int |\int e^{i\Phi(x,\zeta)}\eta(\zeta/\lambda)f(\zeta)d\zeta|^q \psi(x)^q dx)^{1/q} \lesssim \|f\|_p.$$

When $\lambda \to 0$, $\eta(\zeta/\lambda) \to 1$, and the last inequality gives (6.6).

The inequality (6.8) can be proven for much more general phase functions than Φ above, and it has applications to many problems in addition to restriction. The natural and best possible range of exponents in the general case for $n \geq 3$ is

$$1 \leq p \leq \frac{2n+2}{n+3}, q = \frac{n-1}{n+1}p'.$$

It corresponds to the range to which Tomas-Stein restriction leads by interpolation. For $n = 2$ the range can be extended to $1 \leq p \leq 4$. The main part of the proof of (6.8) is for $q = 2, p = \frac{2n+2}{n+3}$. The rest follows by interpolation between this and the trivial case $\|T_\lambda f\|_\infty \lesssim \|f\|_1$. We can write

$$\|T_\lambda f\|^2_{L^2(\mathbb{R}^{n-1})} = \int\int K_\lambda(\xi,\zeta)f(\xi)\overline{f(\zeta)}d\xi d\zeta,$$

where

$$K_\lambda(\xi,\zeta) = \int_{\mathbb{R}^{n-1}} e^{i\lambda(\Phi(x,\xi)-\Phi(x,\zeta))}\Psi(x,\xi)\overline{\Psi(x,\zeta)}dx.$$

Let

$$U_\lambda g(\xi) = \int K_\lambda(\xi,\zeta)g(\zeta)d\zeta.$$

Then

$$\|T_\lambda f\|^2_{L^2(\mathbb{R}^{n-1})} = \int (U_\lambda f)\bar{f}.$$

So we need

$$\|U_\lambda f\|_{L^{p'}(\mathbb{R}^n)} \lesssim \lambda^{-2n/p'}\|f\|_{L^p(\mathbb{R}^n)}.$$

This can be obtained by fairly complicated real and complex interpolation techniques. A gain from going from T_λ to U_λ is that we have now operator which acts on functions in $\mathbb{R}^n$ to functions in $\mathbb{R}^n$ (not $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$ as for T_λ). The formal way from T_λ to U_λ is that the adjoint T^*_λ of T_λ is

$$T^*_\lambda f(\zeta) = \int e^{-i\lambda\Phi(x,\xi)}\overline{\Psi(x,\xi)}f(x)dx,$$

so

$$U_\lambda = T^*_\lambda T_\lambda.$$

A serious problem with U_λ is still that the oscillating factor in its kernel K_λ depends on the variables in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ and cannot have non-degeneracy corresponding to the earlier conditions of non-vanishing Hessian determinant. Here one needs to study the $(n-1) \times n$ matrix

$$(\frac{\partial^2\Phi(x,\xi)}{\partial x_j \partial \xi_k}).$$

What helps is that in many situations it has maximal rank $n-1$, and this is one of the assumptions for a general theorem. This can be used by freezing one coordinate ξ_j and using Fubini arguments or by adding to Φ an auxiliary function $\Phi_0(x,t), t \in \mathbb{R}$, which gives a non-zero Hessian determinant for $\Phi(x,\xi_1,\dots,\xi_{n-1})+$ $\Phi_0(x,\xi_n)$. Then results like Theorem 6.1 can be applied. The many missing details can found in [S] and [So]. subsectionSharp results in the plane In this section we shall prove a sharp $L^p - L^q$-inequality for the operators T_λ in the two-dimensional case. This will solve the restriction conjecture in the plane. These results are due to Carleson and Sjölin from 1972.

First let us observe a corollary to Theorem 6.1: under the assumption (6.2), the operators T_λ of (6.1) satisfy

$$\|T_\lambda f\|_{p'} \lesssim \lambda^{-n/p'} \|f\|_p \text{ for all } f \in L^p(\mathbb{R}^n), \lambda > 0, 1 \le p \le 2. \tag{6.9}$$

This follows readily interpolating (6.3) with the trivial case $\|T_\lambda f\|_\infty \le \|f\|_1$.

We formulate and prove now in the plane a precise result for operators as in (6.7). This kind of results were vaguely discussed before. The variable x will now be a real number and $\xi \in \mathbb{R}^2$. We shall denote derivates with respect to x by $'$ and with respect to ξ_j with subscript ξ_j. So, for example, $\Phi''_{\xi_j}(x,\xi) = \frac{\partial^3 \Phi(x,\xi)}{\partial^2 x \partial \xi_j}$.

Theorem 6.3. Suppose that $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ and $\Psi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ are smooth functions such that Ψ has compact support and

$$\begin{vmatrix} \Phi''_{\xi_1}(x,\xi) & \Phi'_{\xi_1}(x,\xi) \\ \Phi''_{\xi_2}(x,\xi) & \Phi'_{\xi_2}(x,\xi) \end{vmatrix} \neq 0 \text{ for } (x,\xi) \in \operatorname{spt} \Psi. \tag{6.10}$$

Then the operators T^*_λ,

$$T^*_\lambda f(\xi) = \int_{\mathbb{R}} e^{i\lambda\Phi(x,\xi)} \Psi(x,\xi) f(x) dx, \ \lambda > 0, \xi \in \mathbb{R}^2,$$

satisfy

$$\|T^*_\lambda f\|_{L^q(\mathbb{R}^2)} \lesssim \lambda^{-2/q} \|f\|_{L^p(\mathbb{R})} \text{ for all } f \in L^p(\mathbb{R}), \lambda > 0, q = 3p', 1 \le p < 4. \tag{6.11}$$

Remark 6.4. Observe that we we have formulated the theorem for the adjoint operators of the operators T_λ,

$$T_\lambda f(x) = \int_{\mathbb{R}^2} e^{-i\lambda\Phi(x,\xi)} \Psi(x,\xi) f(\xi) dx, \ \lambda > 0, x \in \mathbb{R},$$

that we considered before. The theorem is equivalent with

$$\|T_\lambda f\|_{L^q(\mathbb{R})} \lesssim \lambda^{-2/p'} \|f\|_{L^p} \text{ for all } f \in L^q(\mathbb{R}^2), \lambda > 0, 3q = p', 1 \le p < 4/3.$$

Proof. I shall omit several technical details. They are only routine calculations, except for the last step where we use an inequality for fractional integrals without proving it. What will help is that we have now $q > 4 = 2 \cdot 2$. This allows us to work with

$$T^*_\lambda f(\xi)^2 = \int_{\mathbb{R}^2} e^{i\lambda(\Phi(x,\xi)+\Phi(y,\xi))} \Psi(x,\xi)\Psi(y,\xi) f(x) f(y) dx dy, \ \lambda > 0, \xi \in \mathbb{R}^2.$$

We would like to apply Theorem 6.1 with the phase function $\Phi(x_1,\xi) + \Phi(x_2,\xi)$, $((x_1,x_2),\xi) \in \mathbb{R}^2 \times \mathbb{R}^2$, but the determinant $\det(\frac{\partial^2}{\partial x_j \partial \xi_k}(\Phi(x_1,\xi) + \Phi(x_2,\xi))$ vanishes for $x_1 = x_2$. Computing this determinant, doing a little algebra and applying Taylor's theorem (all quite elementary) one finds that

(6.12)

$$\det(\frac{\partial^2}{\partial x_j \partial \xi_k}(\Phi(x_1,\xi)+\Phi(x_2,\xi))) = \begin{vmatrix} \Phi''_{\xi_1}(x_1,\xi) & \Phi'_{\xi_1}(x_1,\xi) \\ \Phi''_{\xi_2}(x_1,\xi) & \Phi'_{\xi_2}(x_1,\xi) \end{vmatrix} (x_2-x_1) + O(|x_1-x_1|^2).$$

Due to our assumption, we may as before suppose that Ψ has so small support that

$$|\det(\frac{\partial^2}{\partial x_j \partial \xi_k}(\Phi(x_1,\xi)+\Phi(x_2,\xi))| \geq c|x_2-x_1| \text{ when } (x_1,\xi),(x_2,\xi) \in \operatorname{spt}\Psi.$$

Now we would like to make a change of variable in the x variable to get rid of the factor $|x_2-x_1|$. This we obtain with $y = (x_1-x_2, x_1\cdot x_2) =: g(x_1,x_2)$. The Jacobian (the absolute value of the determinant) is $|x_2-x_1|$ as we wanted. Notice that g is only two to one in $\{x : x_1 \neq x_2\}$ but it is one to one in $\{x : x_1 < x_2\}$ and in $\{x : x_2 < x_1\}$. Moreover, $g(x) = g(x')$ if and only if $x = x'$ or $x_1 = x'_2$ and $x_2 = x'_1$, as one checks by direct computation. Setting

$$\tilde{\Phi}(y,\xi) = \Phi(x_1,\xi)+\Phi(x_2,\xi),$$

when $y = g(x)$, we have the well defined function $\tilde{\Phi}$. It is smooth because of the symmetricity of $\Phi(x_1)+\Phi(x_2,\xi)$ with respect to x_1 and x_2. Moreover,

$$\det(\frac{\partial^2}{\partial y_j \partial \xi_k}(\tilde{\Phi}(y,\xi)) \neq 0$$

when $|x_1-x_2|$ is small. Now we have (2 in front comes from the two to one property)

$$T^*_\lambda f(\xi)^2 = 2\int_{\mathbb{R}^2} e^{i\lambda\tilde{\Phi}(y,\xi)}\tilde{\Psi}(y,\xi)F(y)dy,\ \lambda > 0, \xi \in \mathbb{R}^2,$$

where $\tilde{\Psi}(y,\xi) = \Psi(x_1,\xi)\Psi(x_2,\xi)$ is smooth with compact support, and, when $y = g(x)$,

$$F(y) = \frac{f(x_1)\cdot f(x_2)}{|x_1-x_2|}.$$

So we have won by getting a non-vanishing determinant for $\tilde{\Phi}$, but lost by getting a singularity at the diagonal for F. Define r by $2r' = q$. Assuming, as we may, that $q < \infty$, we have then $1 < r < 2$ and we can apply (6.9) getting

$$\int |T^*_\lambda f|^q = \int |(T^*_\lambda f)^2|^{r'} \lesssim \lambda^{-2}(\int |F(y)|^r dy)^{r'/r}.$$

Changing from y to x we have

$$\int |F(y)|^r dy = \frac{1}{2}\int |f(x_1)f(x_2)|^r |x_1-x_2|^{1-r} dx_1 dx_2.$$

To estimate the last integral we use the following Hardy-Littlewood-Sobolev inequality for functions of one variable, see, for example, [S], (31) in Chapter 8:

$$\int |g(x_1)g(x_2)||x_1-x_2|^{-\gamma} dx_1 dx_2 \leq (\int |g|^s)^{2/s},$$

if $1-\frac{1}{s}=\frac{1}{s}-1+\gamma$. We apply this with $g=|f|^r, \gamma=r-1$. Then

$$\int |T_\lambda^* f|^q \lesssim \lambda^{-2}\Big(\int |f|^{rs} dy\Big)^{2r'/(rs)}.$$

The choices of the parameters imply $rs=p$ and $2r'/(rs)=q/p$, and the restriction $1<r<2$ is equivalent to $1<p<4$. Thus the theorem follows. □

If φ is a local parametrization of a curve S and

$$\Phi(x,\xi)=-2\pi(x\cdot\xi_1+\varphi(x)\xi_2)$$

as before in the applications to restriction, then the determinant in the assumptions of Theorem 6.3

$$\begin{vmatrix} \Phi''_{\xi_1}(x,\xi) & \Phi'_{\xi_1}(x,\xi) \\ \Phi''_{\xi_2}(x,\xi) & \Phi'_{\xi_2}(x,\xi) \end{vmatrix} = 4\pi^2\varphi''(x).$$

So the non-vanishing determinant condition means that the curve has non-zero curvature. Recalling the argument '(6.8) implies (6.5)' and checking that the conditions on exponents match we obtain from Theorem 6.3 (recall the formulation in Remark 6.4):

Theorem 6.5. Let S be a smooth compact curve in $\mathbb{R}^2$ with non-vanishing curvature. Then

$$\Big(\int_S |\hat{f}|^q d\sigma^1\Big)^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^2)} \text{ for } f\in L^p(\mathbb{R}^2), 3q=p', 1\le p<4/3.$$

This means in particular that the restriction conjecture 4.3 is valid for the circle S^1.

7. Fourier multipliers

For the beginners there is a nice treatment of the multipliers in [D]. Fefferman's ball example is done in [Gu], [G2] and [S]. [G2], [S] and [So] contain a lot of further material on multipliers.

7.1. Definition and examples.

Let $m\in L^\infty(\mathbb{R}^n)$ be a bounded function. For any function f in $L^2(\mathbb{R}^n)$ we can define the following operator T_m using the Fourier Transform

$$\widehat{T_m f}=m\hat{f}, \text{ that is, } T_m f=(m\widehat{f})^\vee$$

Using Parseval's theorem we get,

$$\|T_m f\|_2=\left\|m\widehat{f}\right\|_2\le \|m\|_\infty \left\|\widehat{f}\right\|_2=\|m\|_\infty \|f\|_2,$$

and therefore T_m is a bounded linear operator from L^2 to L^2 with norm bounded by $\|m\|_\infty$. In fact, this norm is exactly $\|m\|_\infty$ (exercise).

The function $m\in L^\infty(\mathbb{R}^n)$ is said to be an *L^p-multiplier*, $1<p<\infty$, when the operator can be extended to L^p and this extension is bounded from L^p to L^p.

For a measurable set $A\subset\mathbb{R}^n$ we denote

$$T_A=T_{\chi_A}.$$

Let us look at some examples:

Example 7.1. Let m be the sign function in $\mathbb{R}$; $m(x) = -1$ for $x < 0$ and $m(x) = 1$ for $x \geq 1$. Then

$$\widehat{Hf} = -im\hat{f}$$

where H is the Hilbert transform. So $T_m = H$ and m is an L^p-multiplier for all $1 < p < \infty$ by the well-known (but highly non-trivial) results on the Hilbert transform.

Example 7.2. Let $m = m_{a,b}$ be the characteristic function of the interval $[a, b]$ and $S_{a,b} = T_{m_{a,b}}$ the corresponding multiplier operator. This reduces easily to the previous example; in fact, see [D],

$$S_{a,b} = \frac{i}{2}(M_a \circ H \circ M_{-a} - M_b \circ H \circ M_{-b}),$$

where M_a is the multiplication operator: $M_a f(x) = e^{2\pi iax} f(x)$. It follows that $\chi_{[a,b]}$ is an L^p-multiplier for all $1 < p < \infty$. Moreover, its norm is $\leq C_p$ with C_p depending only p. For $a = -R, b = R$, this gives

$$f(x) = \lim_{R\to\infty} \int_{-R}^{R} e^{2\pi ix\xi} \hat{f}(\xi) d\xi \text{ as } R \to \infty \text{ in } L^p \text{ sense.}$$

To prove this, check first that the formula is valid for functions in $\mathcal{S}$ and use the denseness of $\mathcal{S}$.

Example 7.3. As in the previous example, do we also have in $\mathbb{R}^n, n \geq 2$,

$$f(x) = \lim_{R\to\infty} \int_{B(0,R)} e^{2\pi ix\cdot\xi} \hat{f}(\xi) d\xi \text{ as } R \to \infty \text{ in } L^p \text{ sense?}$$

For $p = 2$ we do have. For $p \neq 2$ we don't have. This follows from the fact that in $\mathbb{R}^n$ $\chi_{B(0,1)}$ is an L^p-multiplier if and only if $p = 2$. The proof of this will be the main content of this section.

Observe that the operator norms of $T_{B(0,1)}$ and $T_{B(x,r)}$ for any $x \in \mathbb{R}^n, r > 0$, are equal, check this as an exercise.

Example 7.4. Let $P \subset \mathbb{R}^n$ be a polyhedral domain. Then χ_P is an L^p-multiplier for all $1 < p < \infty$. By definition a polyhedral domain is an intersection of finitely many half-spaces. Thus the claim reduces to showing that the characteristic function of a half-space is an L^p-multiplier. This in turn reduces to the one-dimensional examples above. The details are left as an exercise.

7.2. **Fefferman's example.** The following result is due to C. Fefferman from 1971, [Fe]:

Theorem 7.5. The characteristic function of the the unit ball $B(0, 1)$ in $\mathbb{R}^n, n \geq 2$, is an L^p-multiplier if and only if $p = 2$.

Proof. We shall first consider $n = 2$ and comment on the general case later. The proof is based on Kakeya type constructions. We need a quantitative lemma, Lemma 7.6 on a construction which leads to Besicovitch sets. This is based on the so-called Perron tree. We omit the proof, which can be found in [Gu], [G2] and [S].

For a rectangle R we denote by $\widetilde{R}$ the rectangle obtained by adding translated copies of R and attached along the two shorter sides of R. For example, $\widetilde{R} = [x_1 - l_1, x_1 + 2l_1] \times [x_2, x_2 + l_2]$ if $R = [x_1, x_1 + l_1] \times [x_2, x_2 + l_2], l_1 < l_2$, but we shall consider rectangles in arbitrary directions.

Lemma 7.6. Let $0 < \delta < \frac{1}{2}$. Then there is a measurable set $E \subset \mathbb{R}^2$ and a finite collection of pairwise disjoint rectangles $\{R_k\}$ with the following properties:

(i) $\frac{1}{2} \leq \mathcal{L}^2(E) \leq \frac{3}{2}$,
(ii) $\mathcal{L}^2(E \cap \widetilde{R}_k) \geq \frac{1}{12}\mathcal{L}^2(R_k)$,
(iii) $\mathcal{L}^2(E) \leq \delta \sum \mathcal{L}^2(R_k)$.

First we establish the following general inequality in the spirit that L^p-boundedness for scalar valued operator implies L^p-boundedness for vector valued operators with same norm:

Lemma 7.7. Let $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),\ 1 < p < \infty$, be any bounded linear operator that is $\|Tf\|_p \leq C_p \|f\|_p$ for all $f \in L^p(\mathbb{R}^n)$. Then for every sequence of functions $\{f_j\}_{j=1}^k$ in $L^p(\mathbb{R}^n)$ we have,

$$\left\| \left(\sum_{j=1}^k |Tf_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left(\sum_{j=1}^k |f_j|^2 \right)^{\frac{1}{2}} \right\|_p .$$

Proof. Denote $f = (f_1, \dots, f_k)$ and $Tf = (Tf_1, \dots, Tf_k)$. For $w \in S^{k-1}$ we have by the linearity of T,

$$\int_{\mathbb{R}^n} |w \cdot Tf|^p = \int_{\mathbb{R}^n} |T(w \cdot f)|^p \leq C_p^p \int_{\mathbb{R}^n} |w \cdot f|^p . \tag{7.1}$$

For any $y \in \mathbb{R}^k$ we have,

$$\int_{S^{k-1}} |w \cdot y|^p \, d\sigma^{k-1} w = c\, |y|^p \tag{7.2}$$

where c is independent of y and $|y|$ denotes of course the Euclidean norm on $\mathbb{R}^k$. This follows from the fact that the surface measure σ^{k-1} is rotation invariant, i.e., $\sigma(g(A)) = \sigma(A)$ for all $A \subset S^{k-1}$ and $g \in O(k)$.

Using Fubini's theorem, (7.2) and (7.1) we get

$$c \int_{\mathbb{R}^n} |Tf|^p = \int \int_{\mathbb{R}^n} |w \cdot Tf|^p \, d\sigma^{n-1} w \leq C_p^p \int \int_{\mathbb{R}^n} |w \cdot f|^p \, d\sigma^{n-1} w = C_p^p c \int_{\mathbb{R}^n} |f|^p .$$

This is the required inequality and the proof of the lemma is finished. □

The next lemma associates the multiplier operator of the unit disc to those of the half-spaces:

Lemma 7.8. Assume that the multiplier operator $T = T_B$ of the characteristic function of the unit ball $B = B(0,1) \subset \mathbb{R}^n$ satisfies $\|Tf\|_p \leq C_p \|f\|_p$ for some $p > 2$. Let $\{v_j\}_{j=1}^k$ be a finite sequence of unit vectors in $\mathbb{R}^n$. Let H_j be the half-space,

$$H_j = \{x \in \mathbb{R}^n : v_j \cdot x \geq 0\},$$

and $T_j = T_{H_j}$. Then for any sequence $\{f_j\}_{j=1}^k$ in $L^p(\mathbb{R}^n)$ we have,

$$\left\| \left(\sum_{j=1}^k |T_j f_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left(\sum_{j=1}^k |f_j|^2 \right)^{\frac{1}{2}} \right\|_p .$$

Proof. We assume that $f_j \in \mathcal{S}$, the general case follows by simple approximation. Let B_j^r be the ball of center rv_j and radius $r > 0$. The characteristic functions $\chi_{B_j^r}$ convergence pointwise to χ_{H_j} as $r \to \infty$. Let $T_j^r = (\chi_{B_j^r}\widehat{f})^\vee$. Then also for $f \in \mathcal{S}$, $T_j^r f$ converges to $T_j f$ as $r \to \infty$ both pointwise and in $L^p(\mathbb{R}^n)$. Thus it will suffice to prove that for all $r > 0$,

$$\left\| \left(\sum_{j=1}^k |T_j^r f_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left(\sum_{j=1}^k |f_j|^2 \right)^{\frac{1}{2}} \right\|_p . \tag{7.3}$$

Observe that,

$$T_j^r f(x) = e^{2\pi i r v_j \cdot x} T_r(e^{-2\pi i r v_j \cdot \xi} f)(x),$$

where T_r is the multiplier operator of the disc $B(0,r)$. Set $g_j(\xi) = e^{-2\pi i r v_j \cdot \xi} f_j(\xi)$. We obtain

$$\left\| \left(\sum_{j=1}^k |T_j^r f_j|^2 \right)^{\frac{1}{2}} \right\|_p = \left\| \left(\sum_{j=1}^k |T_r(g_j)|^2 \right)^{\frac{1}{2}} \right\|_p .$$

Recall from Example 7.3 that the operator norms of T_r and T are equal. Therefore Lemma 7.7 yields

$$\left\| \left(\sum_{j=1}^k |T_j^r f_j|^2 \right)^{\frac{1}{2}} \right\|_p = \left\| \left(\sum_{j=1}^k |T_r(g_j)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left(\sum_{j=1}^k |g_j|^2 \right)^{\frac{1}{2}} \right\|_p = C_p \left\| \left(\sum_{j=1}^k |f_j|^2 \right)^{\frac{1}{2}} \right\|_p ,$$

so that (7.3) holds and the proof of the lemma is finished. □

The next lemma tells us how the operators T_j of the previous lemma act on some rectangles. The notation $\tilde{R}$ was introduced before Lemma 7.6.

Lemma 7.9. Let $R \subset \mathbb{R}^2$ be a rectangle whose longer sides are in the direction $v \in S^1$ and let H be the half-plane

$$H = \{x \in \mathbb{R}^2 : v \cdot x \geq 0\}.$$

Then

$$|T_H(\chi_R)| \geq \frac{1}{10} \chi \tilde{R}.$$

Proof. Suppose first that $v = (1,0)$. Denote by T^+ the multiplier for the half line $(0,\infty)$,

$$T^+f(x) = \int_0^\infty \widehat{f}(\xi)e^{2\pi i x\xi}d\xi \ \text{ for } f \in L^2(\mathbb{R}).$$

If $f \in L^2(\mathbb{R})$, Plancherel's theorem gives

$$T^+f(x) = \lim_{\varepsilon\to 0}\int_0^\infty \widehat{f}(\xi)e^{2\pi i(x+i\varepsilon)\xi}d\xi$$

in L^2. Taking $f = \chi_{(-1/2,1/2)}$ we have for $\varepsilon > 0$,

$$\begin{aligned}\int_0^\infty \widehat{f}(\xi)e^{2\pi i(x+i\varepsilon)\xi}d\xi &= \int_{-\infty}^\infty \left(\int_0^\infty e^{-2\pi i y\xi}e^{2\pi i(x+i\varepsilon)\xi}d\xi\right) f(y)dy \\ &= \frac{1}{2\pi i}\int_{-1/2}^{1/2}\frac{1}{(y-x-i\varepsilon)}dy.\end{aligned}$$

by direct elementary estimates the absolute value of the last integral is bounded below for $|x| \leq 1/2$ by some absolute constant. This is enough for us, but one can also check that $1/10$ is OK. It follows that

$$\left|T^+f(x)\right| \geq \frac{1}{10} \quad \text{for} \quad |x| \leq \frac{3}{2}. \tag{7.4}$$

If $f_1, f_2 \in \mathcal{S}$ we have for $F(x_1,x_2) = f_1(x_1)f_2(x_2)$ by Fubini's theorem (recall that $v = (1,0)$),

$$\begin{aligned}T_HF(x_1,x_2) &= \int_H \widehat{F}(\xi)e^{2\pi i x\cdot\xi}d\xi = \\ &= \int_{-\infty}^\infty\int_0^\infty \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)e^{2\pi x_1\xi_1}e^{2\pi i x_2\xi_2}d\xi_1 d\xi_2 \\ &= \left(\int_0^\infty \widehat{f}_1(\xi_1)e^{2\pi x_1\xi_1}d\xi_1\right)\int_{-\infty}^\infty \widehat{f}_2(\xi_2)e^{2\pi i x_2\xi_2}d\xi_2 \\ &= T^+f_1(x_1)f_2(x_2).\end{aligned}$$

Next let R be the rectangle with sides parallel to the axes that is given by a product $\left(-\frac{1}{2},\frac{1}{2}\right)\times(-a,a)$ with $a < \dfrac{1}{2}$. Then,

$$\chi_R(x_1,x_2) = \chi_{(-\frac{1}{2},\frac{1}{2})}(x_1)\chi_{(-a,a)}(x_2).$$

Hence by approximation and the above formula,

$$T_H\chi_R(x_1,x_2) = T^+\chi_{(-\frac{1}{2},\frac{1}{2})}(x_1)\cdot\chi_{(-a,a)}(x_2)$$

and by (7.9) we get that,

$$|T_H\chi_R| \geq \frac{1}{10}\chi_{\widetilde{R}}$$

The same inequality holds for any rectangle R_j with sides parallel to the coordinate axis, either by repeating the proof or changing variables. For an arbitrary rectangle we can rotate and translate to complete the proof. □

Let $T = T_B$ be the multiplier operator of the unit disc $B = B(0,1) \subset \mathbb{R}^2$, i.e., for $f \in L^2(\mathbb{R}^2)$

$$(Tf)^\wedge = \chi_B \widehat{f}.$$

If $f, g \in \mathcal{S}$ we have by Parseval's theorem,

$$\int Tf\bar{g} = \int (Tf)^\wedge \overline{\widehat{g}} = \int \chi_B \widehat{f}\overline{\widehat{g}} = \int \widehat{f}\overline{\chi_B \widehat{g}} = \int \hat{f}\overline{(Tg)^\wedge} = \int f\overline{Tg}.$$

Suppose that T is an L^p multiplier with norm C_p for some $p \neq 2$, and $p' = \dfrac{p}{p-1}$. By duality and Hölder's inequality,

$$\begin{aligned}\|Tf\|_{p'} &= \sup_{\|g\|_p \le 1, g \in \mathcal{S}} \left| \int Tf\bar{g} \right| = \sup_{\|g\|_p \le 1, g \in \mathcal{S}} \left| \int f\overline{Tg} \right| \\ &\le \sup_{\|g\|_p \le 1, g \in \mathcal{S}} \|f\|_{p'} \|Tg\|_p \le \sup_{\|g\|_p \le 1, g \in \mathcal{S}} \|f\|_{p'} C_p \|g\|_p \le C_p \|f\|_{p'} .\end{aligned}$$

Hence T is also an $L^{p'}$-multiplier. Therefore without loss of generality we can assume that $p > 2$.

Let E and R_j be as in Lemma 7.6, $f_j = \chi_{R_j}$, let $v_j \in S^1$ be the directions of the longer sides of R_j and let T_j be the half-plane multiplier related to v_j as in Lemma 7.8.

First notice that by Lemmas 7.3 and 7.6 we have with $c_0 = 1/100$,

$$\begin{aligned}\int_E \left(\sum_{j=1}^k |T_j f_j|^2 \right) &= \sum_{j=1}^k \int_E |T_j f_j|^2 \ge \sum_{j=1}^k c_0 \int_E \chi_{\widetilde{R_j}} \\ &= c_0 \sum_{j=1}^k \mathcal{L}^2(E \cap \widetilde{R_j}) \ge c_0^2 \sum_{j=1}^k \mathcal{L}^2(R_j).\end{aligned}$$

By Hölders inequality for the exponents $\frac{p}{2}$ and $(\frac{p}{2})' = \frac{p}{p-2}$ and Lemma 7.8,

$$\begin{aligned}\int_E \left(\sum_{j=1}^k |T_j f_j|^2 \right) &\le \left(\int_{\mathbb{R}^2} \left(\sum_{j=1}^k |T_j f_j|^2 \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \mathcal{L}^2(E)^{\frac{p-2}{p}} \\ &= \left\| \left(\sum_{j=1}^k |T_j f_j|^2 \right)^{\frac{1}{2}} \right\|_p^2 \mathcal{L}^2(E)^{\frac{p-2}{p}} \le C_p \left\| \left(\sum_{j=1}^k |f_j|^2 \right)^{\frac{1}{2}} \right\|_p^2 \mathcal{L}^2(E)^{\frac{p-2}{p}} \\ &= C_p \mathcal{L}^2(E)^{\frac{p-2}{p}} \left(\sum_{j=1}^k \mathcal{L}^2(R_j) \right)^{\frac{2}{p}}\end{aligned}$$

The last equality follows because the rectangles R_j are disjoint. Combining the previous estimates,

$$c_0^2 \sum_{j=1}^k \mathcal{L}^2(R_j) \le c_0^2 C_p \mathcal{L}^2(E)^{\frac{p-2}{p}} \left(\sum_{j=1}^k \mathcal{L}^2(R_j)\right)^{\frac{2}{p}} \le c_0^2 C_p \delta^{\frac{p-2}{p}} \sum_{j=1}^k \mathcal{L}^2(R_j),$$

which is a contradiction for sufficiently small δ.

This completes the proof for $n = 2$. For $n > 2$ one can proceed in the same way and at the end use for the the functions f_j,

$$f_j(x_1, \dots, x_n) = \chi_{R_j}(x_1, x_2) f(x_3, \dots, x_n),$$

where f is a fixed function on $\mathbb{R}^{n-2}$. One can also prove and use a Fubini-type result stating that if m is an L^p-multiplier on $\mathbb{R}^{m+n}$, then for almost every $\xi \in R^m$, $\eta \mapsto m(\xi, \eta)$ is an L^p-multiplier with norm bounded by that of m. For this see [G1], Theorem 2.5.16. ☐

7.3. **Bochner-Riesz multipliers.** I don't give here proofs. For them and further results and comments, see [D], [G2], [S] and [So].

We know now that we don't have the convergence asked for in Example 7.3 if $n \geq 2$ and $p \neq 2$. But what about some modified type of convergence, for example,

$$f(x) = \lim_{R \to \infty} \int_{B(0,R)} (1 - \frac{|\xi|}{R}) e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi \text{ as } R \to \infty \text{ in } L^p \text{ sense?}$$

This is analogous to some classical facts for Fourier series: it is easier to get the convergence for instance in Cesàro sense, leading to the Fejér kernel, than for the usual Fourier partial sums, see, e.g., [D]. So we are requesting about results for the multiplier $(1 - |\xi|)_+$, or equivalently for $m(\xi) = (1 - |\xi|^2)_+$, instead of the ball multiplier. Here $a_+ = \max\{a, 0\}$. Raising m to small power $\delta > 0$ we get closer to the characteristic function of the unit ball.

Definition 7.10. The Bochner-Riesz multiplier with parameter $\delta > 0$ is defined by

$$m_\delta(\xi) = (1 - |\xi|^2)_+^\delta.$$

The corresponding multiplier operator is S_δ;

$$S_\delta f = (m_\delta \hat{f})^\vee.$$

For $f \in \mathcal{S}$ we have

$$S_\delta f = K_\delta * f.$$

The kernel K_δ can be computed from the formula for the Fourier transform of a radial function with the aid of some Bessel function identities. It is

$$K_\delta(x) = c(\delta)|x|^{-n/2-\delta} J_{n/2+\delta}(2\pi|x|),$$

see [D] or [S]. From the properties of the Bessel functions it follows that K_δ is bounded and its aymptotic behaviour at infinity is,

$$K_\delta(x) \approx E(x)|x|^{-n/2-\delta-1/2},$$

where E is a bounded linear combination of trigonometric functions. Consequently, $K_\delta \in L^p(\mathbb{R}^n)$ if and only if $p > \frac{2n}{n+1+2\delta}$ This implies that m_δ is not an L^p-multiplier if $p \leq \frac{2n}{n+1+2\delta}$. By duality, neither is it when $p \geq \frac{2n}{n-1-2\delta}$. The *Bochner-Riesz conjecture* believes that these are the only restrictions:

Conjecture 7.11. m_δ is an L^p-multiplier if and only if

$$\frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta}.$$

Notice that the above condition is equivalent to

$$|\frac{1}{p} - \frac{1}{2}| < \frac{2\delta+1}{2n}.$$

Again, this conjecture is open for $n \geq 3$ and true for $n = 2$. In $\mathbb{R}^2$ it is very close to the restriction conjecture and was also proved by Carleson and Sjölin; Theorem 6.3 which gave the restriction conjecture gives also this, see [S]. Tao proved in 1999 in [Tao] that the Bochner-Riesz conjecture implies restriction conjecture, there are also some partial results in the opposite direction, see Tao's paper.

The following theorem is rather close to the best that is known:

Theorem 7.12. m_δ is an L^p-multiplier if

$$\frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta},$$

and

$$\delta > \frac{n-1}{2(n+1)}.$$

The proof is easy if $\delta > \frac{n-1}{2}$, because then $K_\delta \in L^1$. It is simpler than in the full range for

$$|\frac{1}{p} - \frac{1}{2}| < \frac{\delta}{n-1},$$

see [D] for that. For the full range one can use restriction Tomas-Stein restriction theorem. Here is a skecth of the proof following a lecture of Ana Vargas:

We can write

$$m_\delta(\xi) = \sum_{k=0}^{\infty} 2^{-k\delta}\varphi_k(|\xi|),$$

where the functions φ_k are smooth, $\operatorname{spt}\varphi_k \subset (1-2^{1-k}, 1-2^{-2-k})$ for $k \geq 1$, and $|\varphi_k^{(j)}(t)| \leq C_j 2^{kj}$ for $t \in \mathbb{R}, j = 0, 1, 2, \dots$. Let T_k be the multiplier operator

$$\widehat{T_k f}(\xi) = \varphi_k(|\xi|)\hat{f}(\xi).$$

Then

$$S_\delta = \sum_{k=0}^{\infty} 2^{-k\delta} T_k.$$

The kernel of T_k (the Fourier transform of $\varphi_k(|\xi|)$) decays very fast for $|x| > 2^k$, which implies that for the boundedness of T_k it suffices to consider functions f supported on $B(0, 2^k)$. For such an f,

$$T_k f(x) = \int_{1-2^{1-k}}^{1-2^{-2-k}} \int_{S^{n-1}} e^{2\pi i r x\cdot\xi} \hat{f}(r\xi)\varphi_k(r) d\sigma^{n-1}\xi r^{n-1} dr.$$

Hence by Minkowski's integral inequality,

$$\|T_k f\|_{L^q(\mathbb{R}^n)} \lesssim \int_{1-2^{1-k}}^{1-2^{-2-k}} \| \int_{S^{n-1}} e^{2\pi i r x\cdot\xi} \hat{f}(r\xi) d\sigma^{n-1}\xi \|_{L^q(\mathbb{R}^n)} r^{n-1} dr.$$

Theorem 4.4 yields then with $q = 2\frac{n+1}{n-1}$,

$$\|T_k f\|_{L^q(\mathbb{R}^n)} \lesssim \int_{1-2^{1-k}}^{1-2^{-2-k}} \|\hat{f}(r\cdot)\|_{L^2(S^{n-1})} r^{n-1} dr.$$

From this we obtain using Schwartz's inequality, the fact that $r \approx 1$ and Plancherel's theorem,

$$\|T_k f\|_{L^q(\mathbb{R}^n)} \lesssim (\int_{1-2^{1-k}}^{1-2^{-2-k}} \|\hat{f}(r\cdot)\|_{L^2(S^{n-1})}^2 r^{n-1} dr)^{1/2} 2^{-k/2} \leq \|f\|_{L^2(\mathbb{R}^n)} 2^{-k/2}.$$

Since $\operatorname{spt} f \subset B(0, 2^k)$, we get

$$\|T_k f\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-k/2} 2^{kn\frac{q-2}{2q}} \|f\|_{L^q(\mathbb{R}^n)} = 2^{k\frac{n-1}{2(n+1)}} \|f\|_{L^q(\mathbb{R}^n)}.$$

Recalling now that $S_\delta = \sum_{k=0}^{\infty} 2^{-k\delta} T_k$ and that $\delta > \frac{n-1}{2(n+1)}$, we see that S_δ is bounded $L^q \to L^q$ for $q = 2\frac{n+1}{n-1}$. The general case follows from this by interpolation.

Stein uses in [S] again stationary phase. The formulation is a little different;

$$\delta > \frac{n-1}{2(n+1)}$$

is replaced by

$$1 \leq p \leq \frac{2(n+1)}{n+3} \text{ or } \frac{2(n+1)}{n-1} \leq p < \infty,$$

but presumably these are essentially equivalent. The key lemma is

Lemma 7.13. Let ψ be a smooth function with compact support in $\mathbb{R}^n$. Define

$$G_\lambda f(x) = \int e^{\lambda|x-y|} \psi(x-y) f(y) dy, x \in \mathbb{R}^n, \lambda > 0.$$

Then for $1 \leq p \leq \frac{2(n+1)}{n+3}$ and $\lambda > 0$,

$$\|G_\lambda f\|_p \lesssim \lambda^{-n/p'} \|f\|_p.$$

A difference to the earlier case is that the phase function $|x-y|$ is not smooth. To overcome this one can consider

$$\tilde{G}_\lambda f(x) = \int e^{\lambda|x-y|} \tilde{\psi}(x-y) f(y) dy,$$

where the support of $\tilde{\psi}$ does not meet the origin. For this and other details, see [S].

7.4. **Summary of conjectures.** I collect here the conjectures we have discussed and some relations between them:

(1) Kakeya conjecture 5.1:
Every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension n.

(2) Kakeya maximal conjecture 5.4:

$$\|f_\delta^*\|_{L^n(S^{n-1})} \leq C_\varepsilon \delta^{-\varepsilon}\|f\|_n \text{ for all } \varepsilon > 0, f \in L^n(\mathbb{R}^n).$$

(3) Restriction conjecture 4.3:

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \leq C_q\|f\|_{L^\infty(S^{n-1})} \text{ (or } \leq C_q\|f\|_{L^q(S^{n-1})}) \text{ for } q > 2n/(n-1).$$

(4) Bochner-Riesz conjecture 7.12:
m_δ is an L^p-multiplier if and only if

$$\frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta}.$$

We proved that (2) implies (1) in Theorem 5.5 and (3) implies (2) in Theorem 5.12. As mentioned above (4) implies (3) by [Tao]. All these conjectures are true in $\mathbb{R}^2$.

8. (n,k) Besicovitch sets

What can we say if we replace in the definition of Besicovitch sets the line segments with pieces of k-dimensional planes?

We denote by $G(n.k)$ the space of k-dimensional linear subspaces of $\mathbb{R}^n$. It is a compact metric space with, for example, the metric

$$d(V,W) = \|P_V - P_W\|,$$

where $P_V : \mathbb{R}^n \to V$ is the orthogonal projection and $\|L\|$ is any norm for linear maps $L : \mathbb{R}^n \to \mathbb{R}^n$. We denote $\gamma_{n,k}$ the unique orthogonally invariant Borel probability measure on $G(n,k)$. It can be defined by

$$\gamma_{n,k}(A) = \theta_n(\{g \in O(n) : g(V_0) \in A,\ A \subset G(n,k)\}),$$

where θ_n is the Haar probability measure on the orthogonal group $O(n)$ and $V_0 \in G(n,k)$ is any fixed k-plane. For $k=1$ and $k=n-1$ we can reduce this measure to the surface measure on S^{n-1}; denoting $L_v = \{tv : t \in \mathbb{R}\}$,

$$\gamma_{n,1}(A) = c(n)\sigma^{n-1}(\{v \in S^{n-1} : L_v \in A\}),\ A \subset G(n,1),$$
$$\gamma_{n,n-1}(A) = c(n)\sigma^{n-1}(\{v \in S^{n-1} : L_v^\perp \in A\}),\ A \subset G(n,n-1).$$

Definition 8.1. A set $B \subset \mathbb{R}^n$ is said to be an (n,k) Besicovitch set if $\mathcal{L}^n(B) = 0$ and for every $V \in G(n,k)$ there is $a \in \mathbb{R}^n$ such that $\mathcal{H}^k(B \cap (V+a)) > 0$.

Here $\mathcal{H}^k$ is the k-dimensional Haudorff measure whose restriction to a k-plane is just the Lebesgue measure of that plane. We could as well ask that $B \cap (V + a)$ contains a unit k ball as in the case $k = 1$. The existence questions would probably be equivalent, though I am not sure.

The first question is: do they exist if $k > 1$? The first result on this was proved by Marstrand in 1979 in [M1]:

Theorem 8.2. There are no $(3, 2)$ Besicovitch sets. More precisely, if $E \subset \mathbb{R}^3$ and $\mathcal{L}^3(E) = 0$, then for almost all $V \in G(3, 2)$, $\mathcal{H}^2(E \cap (V + a)) = 0$ for all $a \in \mathbb{R}^3$.

Proof. Clearly we can assume that $E \subset B(0, 1/2)$. Denote for $v \in S^2$ and $A \subset B(0, 1)$,

$$f(A, v) = \sup\{\mathcal{H}^2(A \cap (L_v^\perp + a)) : a \in \mathbb{R}^3\}.$$

We shall prove that

$$(\int^* f(A, V) d\gamma_{3,2} V)^2 \lesssim \mathcal{L}^3(A), \tag{8.1}$$

where $\int^*$ is the upper integral. The theorem clearly follows from this. Obviously it suffices to prove (8.1) for open sets A. It is easy to check that if $B_i \subset \mathbb{R}^3$ is an increasing sequence of Borel sets with $B = \cup_i B_i$, then $f(B, V) \leq \liminf_{i\to\infty} f(B_i, V)$. Therefore it is enough to prove (8.1) for disjoint finite unions of cubes of the same side-length (when adding new smaller cubes split the earlier ones to match the size). Thus let $B = \cup_{i=1}^m Q_j \subset B(0, 1)$ where the Q_j's are disjoint cubes of side-length δ.

For every $v \in S^2$ the function $a \mapsto \mathcal{H}^2(B \cap (L_v^\perp + a))$ attains its supremum for some $a \in \mathbb{R}^3$; except the v's orthogonal to coordinate planes it is a continuous function of a and for these six exceptional vectors it takes only finitely many values. Choose for every $v \in S^2$ some $a \in \mathbb{R}^3$ such that with $A(v) = L_v^\perp + a$ we have $F(B, v) = \mathcal{H}^2(B \cap A(v))$. Clearly this choice can be made so that the function A is a Borel function.

We can now estimate using Schwartz's inequality and Fubini's theorem,

$$
\begin{aligned}
&(\int f(B,v)d\sigma^2 v)^2 = (\int \mathcal{H}^2(B\cap A(v))d\sigma^2 v)^2 \\
&= (\sum_{j=1}^m \int \mathcal{H}^2(Q_j\cap A(v))d\sigma^2 v)^2 \leq m\sum_{j=1}^m (\int \mathcal{H}^2(Q_j\cap A(v))d\sigma^2 v)^2 \\
&= m\sum_{j=1}^m \int_{S^2\times S^2} \mathcal{H}^2\times\mathcal{H}^2((Q_j\times Q_j)\cap(A(v)\times A(w)))d(\sigma^2\times\sigma^2)(v,w) \\
&= m\int_{S^2\times S^2} \mathcal{H}^2\times\mathcal{H}^2(\cup_{j=1}^m(Q_j\times Q_j)\cap(A(v)\times A(w)))d(\sigma^2\times\sigma^2)(v,w) \\
&\leq m\int_{S^2\times S^2} \mathcal{H}^2\times\mathcal{H}^2(\{(x,y)\in A(v)\times A(w) : |x|,|y|\leq 1, |x-y|\leq\sqrt{3}\delta\})d(\sigma^2\times\sigma^2)(v,w) \\
&\leq m\int_{S^2\times S^2}\int_{B(0,1)\cap A(v)} \mathcal{H}^2(B(x,\sqrt{3}\delta)\cap A(w))d\mathcal{H}^2 x d(\sigma^2\times\sigma^2)(v,w) \\
&\leq 3\pi\delta^2 m\int_{S^2\times S^2} \mathcal{H}^2(\{x\in B(0,1)\cap A(v) : dist(x,A(w))\leq\sqrt{3}\delta\})d(\sigma^2\times\sigma^2)(v,w)
\end{aligned}
$$

.

We estimate the last integrand by elementary geometry. For this we may assume $v\neq\pm w$ and that the planes $A(v)$ and $A(w)$ go through the origin. Then $A(v)\neq A(w)$ and $A(v)$ and $A(w)$ intersect along a line $L\in G(3,1)$. Denote by $2\alpha(v,w)$ the angle between v and w. Then if $x\in A(v)\cap B(0,1)$ and $dist(x,A(w))\leq\sqrt{3}\delta$, we must have $|x|\leq\frac{\sqrt{3}\delta}{\sin(\alpha(v,w))}$. This implies that our set is contained in a rectangle with sidelengths $\frac{2\sqrt{3}\delta}{\sin(\alpha(v,w))}$ and 2. This gives

$$\mathcal{H}^2(\{x\in B(0,1)\cap A(v) : dist(x,A(w))\leq\sqrt{3}\delta\})\leq\frac{4\sqrt{3}\delta}{\sin(\alpha(v,w))},$$

and

$$(\int f(B,v)d\sigma^2 v)^2\leq 3\pi\delta^2 m\int\frac{4\sqrt{3}\delta}{\sin(\alpha(v,w))}d(\sigma^2\times\sigma^2)(v,w).$$

For any fixed $w\in S^2$,

$$\int\sin(\alpha(v,w))^{-1}d\sigma^2 v\approx\int_{B^2(0,1)}|x|^{-1}dt\approx 1.$$

Combining these we conclude

$$(\int f(B,v)d\sigma^2 v)^2\lesssim m\delta^3=\mathcal{L}^3(B),$$

as required. □

The following result was proved by Falconer in [F1]:

Theorem 8.3. There are no (n,k) Besicovitch sets for $k > n/2$. More precisely, if $k > n/2$ and $E \subset \mathbb{R}^n$ with $\mathcal{L}^n(E) = 0$, then for almost all $V \in G(n,k)$,

$$\mathcal{H}^k(E \cap (V+a)) = 0 \text{ for all } a \in \mathbb{R}^n.$$

Proof. We shall use the following formula, say for non-negative Borel functions f:

$$\int_{\mathbb{R}^n} f d\mathcal{L}^n = c(n,k)\int_{G(n,k)}\int_{V^\perp} |x|^k f(x) d\mathcal{H}^k x d\gamma_{n,k}V. \tag{8.2}$$

To prove this, integrate the right hand side in the spherical coordinates of $V^\perp$:

$$\begin{aligned}
&\int_{G(n,k)}\int_{V^\perp} |x|^k f(x) d\mathcal{H}^k x d\gamma_{n,k}V \\
&= \int_{G(n,k)}\int_0^\infty \int_{V^\perp \cap S^{n-1}} r^k f(rv) d\sigma^{k-1} v r^{n-k-1} dr d\gamma_{n,k}V \\
&= \int_0^\infty r^{n-1}\int_{G(n,k)}\int_{V^\perp \cap S^{n-1}} f(rv) d\sigma^{k-1} v d\gamma_{n,k}V dr.
\end{aligned}$$

For non-negative Borel functions on S^{n-1},

$$\int_{G(n,k)}\int_{V^\perp \cap S^{n-1}} g(v) d\sigma^{k-1} v d\gamma_{n,k}V = c(n,k)\int_{S^{n-1}} g d\sigma^{n-1},$$

because the left hand side defines an orthogonally invariant measure on S^{n-1} and such a measure is unique up to multiplication by a constant. Thus

$$\int_{G(n,k)}\int_{V^\perp} |x|^k f(x) d\mathcal{H}^k x d\gamma_{n,k}V = \int_0^\infty r^{n-1}\int_{S^{n-1}} f(rv) d\sigma^{n-1} v dr = \int_{\mathbb{R}^n} f d\mathcal{L}^n.$$

Suppose now $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let $V \in G(n,k)$. If $\xi \in V^\perp$, then, writing for a moment $x = x_V + x'_V, x_V \in V, x'_V \in V^\perp$, we have by Fubini's theorem,

$$\hat{f}(\xi) = \int_{V^\perp} e^{-2\pi i \xi \cdot x'_V}\int_{V + x'_V} f d\mathcal{H}^k dx'_V = \widehat{F_V}(\xi).$$

Thus by (8.2) and Schwartz's inequality,

$$\begin{aligned}
&\int_{G(n,k)}\int_{\{\xi \in V^\perp : |\xi| \geq 1\}} |\widehat{F_V}(\xi)| d\mathcal{H}^k \xi d\gamma_{n,k}V = \int_{G(n,k)}\int_{\{\xi \in V^\perp : |\xi| \geq 1\}} |\hat{f}(\xi)| d\mathcal{H}^k \xi d\gamma_{n,k}V \\
&= c(n,k)\int_{\{\xi \in \mathbb{R}^n : |\xi| \geq 1\}} |\hat{f}(\xi)||\xi|^{-k} d\xi \leq c(n,k)\left(\int |\hat{f}(\xi)|^2 d\xi\right)^{1/2}\left(\int_{\{\xi \in \mathbb{R}^n : |\xi| \geq 1\}} |\xi|^{-2k} d\xi\right)^{1/2} \\
&= c'(n,k)\|f\|_2,
\end{aligned}$$

where $c'(n,k) < \infty$ since $2k > n$. As $\|\hat{f}\|_\infty \leq \|f\|_1$ and $k < n$, we have also

$$\begin{aligned}
&\int_{G(n,k)}\int_{\{\xi \in V^\perp : |\xi| \leq 1\}} |\widehat{F_V}(\xi)| d\mathcal{H}^k \xi d\gamma_{n,k}V = \\
&c(n,k)\int_{\{\xi \in \mathbb{R}^n : |\xi| \leq 1\}} |\hat{f}(\xi)||\xi|^{-k} d\xi \lesssim \|f\|_1.
\end{aligned}$$

So we see that for almost all $V \in G(n,k)$, $\widehat{F_V} \in \mathcal{L}^1(V^\perp)$. By Fubini's theorem also $F_V \in \mathcal{L}^1(V^\perp)$ for all $V \in G(n,k)$. Thus Fourier inversion formula implies $\|F_V\|_{L^\infty(V^\perp)} \leq \|\widehat{F_V}\|_{L^1(V^\perp)}$ for almost all $V \in G(n,k)$. Consequently,

$$\int_{G(n,k)} \|F_V\|_{L^\infty(V^\perp)} d\gamma_{n,k}V \leq \int_{G(n,k)} \int_{V^\perp} |\widehat{F_V}(\xi)| d\mathcal{H}^k\xi d\gamma_{n,k}V \lesssim \|f\|_1 + \|f\|_2. \tag{8.3}$$

Now we return to E. Take a sequence U_i of open sets such that $E \subset U_i$ and $\mathcal{L}^n(U_i) \to 0$. We apply (8.3) to the characteristic functions $f_i := \chi_{U_i}$ of U_i. Let $F_{i,V}$ be the related functions on $V^\perp$; $F_{i,V}(a) = \mathcal{H}^k(U_i \cap (V+a))$. The key observation now is that

$$\sup\{\mathcal{H}^k(U_i \cap (V+a)) : a \in V^\perp\} = \|F_{i,V}\|_{L^\infty(V^\perp)}.$$

This is easy to see: if M is the supremum on the left hand side and $t < M$, pick $a \in V^\perp$ such that $\mathcal{H}^k(U_i \cap (V+a)) > t$. Then $\mathcal{H}^k(K+a) > t$ for some compact subset K of V such that $K + a \subset U_i \cap (V+a)$. Then $K + b \subset U_i \cap (V+b)$ for b in some neighborhood of a, and the observation follows. Combining this with (8.3) we conclude that

$$\begin{aligned} &\int^* \sup\{\mathcal{H}^k(E \cap (V+a)) : a \in V^\perp\} d\gamma_{n,k}V \\ &\leq \liminf_{i\to\infty} \int \sup\{\mathcal{H}^k(U_i \cap (V+a)) : a \in V^\perp\} d\gamma_{n,k}V \\ &\lesssim \liminf_{i\to\infty} (\mathcal{L}^n(U_i) + \mathcal{L}^n(U_i)^{1/2}) = 0. \end{aligned}$$

This proves the theorem. □

The above proof shows that for almost all $V \in G(n,k)$ the functions F_V agree almost everywhere with continuous functions. It can be developed to give more information about the differentiability properties of these functions for $f \in L^p$, see [F1].

It is an open question whether there exist (n,k) Besicovitch sets for any $k > 1$. With considerable effort the above simple results can be improved. This is due to Bourgain from [B] and the best known information is that there exist no (n,k) Besicovitch sets if $2^{k-1} + k \geq n$.

9. HAUSDORFF DIMENSION OF BESICOVITCH SETS

Since we cannot solve the Kakeya conjecture, we could at least try to find lower bounds for the Hausdorff dimension of Besicovitch sets. The trivial one is 1. We have also the lower bound 2. We proved it in the plane and essentially the same proof gives it in higher dimensions. In fact, it follows immediately combining Remarks 5.6 and 5.8. One can also argue that the projection of a Besicovitch set on any 2-plane is essentially a Besicovitch set (the line segments could have length less that 1) and since projections don't increase dimension, we get the above statement.

9.1. Bourgain's bushes and lower bound $(n+1)/2$.

Next we shall derive lower bound $\frac{n+1}{2}$. First we observe that the proof of Theorem 5.5 gives the following, check this as an exercise:

Theorem 9.1. Suppose that for some $1 \le p \le q < \infty$ and $\beta > 0$ there is a positive number C such that

$$\sigma^{n-1}(\{e \in S^{n-1} : (\chi_E)^*_\delta(e) > \lambda\}) \le C(\lambda^{-1}\delta^{-\beta}\mathcal{L}^n(E)^{1/p})^q \tag{9.1}$$

for all Lebesgue measurable sets $E \subset \mathbb{R}^n$ and all positive numbers δ and λ. Then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $n - \beta p$.

Our next plan is to verify (9.1) for $p = (n+1)/2, q = n+1$ and $\beta = n-1$ to get the lower bound $(n+1)/2$ for the Hausdorff dimension of Besicovitch sets. Before doing this let us contemplate a little what (9.1) means. It is a restricted weak type inequality (restricted since it only deals with characteristic functions) which would follow immediately by Chebyshev's inequality from the corresponding strong type inequality (if we knew it):

$$\|f^*_\delta\|_{L^q(S^{n-1})} \lesssim \delta^{-\beta}\|f\|_p.$$

The converse is not true, but if we have restricted weak type inequalities for pairs (p_1, q_1) and (p_2, q_2) we have the strong type inequality for the appropriate pairs (p, q) between (p_1, q_1) and (p_2, q_2) by interpolation results of Stein and Weiss, see [SW] or [G1]. Recall the Kakeya maximal conjecture 5.1:

$$\|f^*_\delta\|_{L^n(S^{n-1})} \le C_\varepsilon \delta^{-\varepsilon}\|f\|_n \text{ for all } \varepsilon > 0.$$

Interpolating this with the trivial estimate $\|f^*_\delta\|_{L^\infty(S^{n-1})} \le C_\varepsilon \delta^{1-n}\|f\|_1$ gives the equivalent conjecture

$$\|f^*_\delta\|_{L^q(S^{n-1})} \le C_\varepsilon \delta^{-(n/p-1+\varepsilon)}\|f\|_p \text{ for all } \varepsilon > 0, 1 \le p \le n, q = (n-1)p'.$$

In the next theorem we shall prove the restricted weak type version of this corresponding to $p = (n+1)/2, q = n+1$.

Theorem 9.2. There is a positive number $C(n)$ such that

$$\sigma^{n-1}(\{e \in S^{n-1} : (\chi_E)^*_\delta(e) > \lambda\}) \le C(n)\delta^{1-n}\lambda^{-(n+1)}\mathcal{L}^n(E)^2 \tag{9.2}$$

for all Lebesgue measurable sets $E \subset \mathbb{R}^n$ and all positive numbers δ and λ. In particular, the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $(n+1)/2$.

Proof. The inequality (9.2) means that the assumption (9.1) holds with $p = \frac{n+1}{2}, q = n+1$ and $\beta = \frac{n-1}{n+1}$ so that the statement about Besicovitch sets follows from Theorem 9.1.

Let $S^{n-1}_+ = \{e \in S^{n-1} : e_n > 1/2\}$ and

$$A = \{e \in S^{n-1}_+ : (\chi_E)^*_\delta(e) > \lambda\}.$$

To prove (9.2) it is enough to estimate the measure of A. We can choose a maximal δ-separated set $\{e_1, \dots, e_N\} \subset A$ such that $N \gtrsim \delta^{1-n}\sigma^{n-1}(A)$ and tubes $T_j = T^\delta_{e_j}(a_j)$ for which

$$\mathcal{L}^n(E \cap T_j) > \lambda \mathcal{L}^n(T_j) \approx \lambda\delta^{n-1}. \tag{9.3}$$

To prove (9.2) it suffices then to show that

$$\mathcal{L}^n(E) \gtrsim \sqrt{N}\delta^{n-1}\lambda^{\frac{n+1}{2}}. \tag{9.4}$$

Let m be the smallest integer such that every point of E belongs to at most m tubes T_j. This means that

$$\sum_j \chi_{E\cap T_j} \leq m \tag{9.5}$$

and there is $x \in E$ which belongs at least to m tubes T_j. Integration of (9.5) over E gives by (9.3) that

$$\mathcal{L}^n(E) \gtrsim \sum_j m^{-1}\mathcal{L}^n(E \cap T_j) \gtrsim m^{-1}N\lambda\delta^{n-1}. \tag{9.6}$$

To make use of x assume that it belongs to the first m tubes T_j; $x \in T_j$ for $j = 1, \dots, m$. Let c be a positive constant depending only n such that

$$\mathcal{L}^n(B(x, c\lambda) \cap T^\delta_e(a)) \leq \frac{\lambda}{2}\mathcal{L}^n(T^\delta_e(a))$$

for every $e \in S^{n-1}, a \in \mathbb{R}^n$; the existence of such a constant is an easy exercise. Then by (9.3) for $j = 1, \dots, m$,

$$\mathcal{L}^n(E \cap T_j \setminus B(x, c\lambda)) \geq \frac{\lambda}{2}\mathcal{L}^n(T_j) \approx \lambda\delta^{n-1}. \tag{9.7}$$

By simple elementary plane geometry there is an absolute constant $b \geq c$ such that for any $e, e' \in S^{n-1}_+, a_1, a_2 \in \mathbb{R}^n$,

$$\operatorname{diam}(T^\delta_e(a_1) \cap T^\delta_{e'}(a_2)) \leq \frac{b\delta}{|e - e'|}. \tag{9.8}$$

Let $e'_1, \dots, e'_{m'}$ be a maximal $\frac{b\delta}{c\lambda}$-separated subset of $e_1, \dots, e_m$. Here $\frac{b\delta}{c\lambda} \geq \delta$, when we assume, as we of course may, that $\lambda \leq 1$. The balls $B(e'_k, \frac{2b\delta}{c\lambda}), k = 1, \dots, m'$, cover the disjoint balls $B(e_j, \delta/3), j = 1, \dots, m$. Thus

$$m\delta^{n-1} \approx \sigma^{n-1}(\cup_{j=1}^m B(e_j, \delta/3)) \leq \sigma^{n-1}(\cup_{k=1}^{m'} B(e'_k, \frac{2b\delta}{c\lambda})) \lesssim m'(\delta/\lambda)^{n-1},$$

whence $m' \gtrsim \lambda^{n-1}m$. By (9.8) the sets $E \cap T'_k \setminus B(x, c\lambda), k = 1, \dots, m'$, ($T'_k$ corresponds to e'_k) are disjoint. Therefore by (9.7),

$$\mathcal{L}^n(E) \gtrsim \lambda\delta^{n-1}m' \gtrsim \lambda^n\delta^{n-1}m. \tag{9.9}$$

Now both inequalities (9.6) and (9.9) hold. Consequently

$$\mathcal{L}^n(E) \gtrsim \max\{\lambda^n\delta^{n-1}m, m^{-1}N\lambda\delta^{n-1}\} \geq \sqrt{(\lambda^n\delta^{n-1}m)(m^{-1}N\lambda\delta^{n-1})} = \sqrt{N}\delta^{n-1}\lambda^{\frac{n+1}{2}}.$$

and (9.4) follows. □

The above theorem is due to Drury from 1983; he proved L^p-estimates for the X-ray transform $Tf(L) = \int_L f, L \subset \mathbb{R}^n$ a line, which are essentially the same as the Kakeya estimates. The bound $(n+1)/2$ is a bit annoying as it doesn't agree with the optimal bound 2 in the plane. The above method with 'bushes', bunches of tubes containing a common point, is due to Bourgain from [B]. In fact, Bourgain proved sharper Kakeya estimates with better bounds for the Hausdorff dimension of Besicovitch sets. This method also lead to the non-existence of (n, k) Besicovitch sets for $2^{k-1} + k \geq n$ mentioned in the previous section, and to the first partial results on the restriction conjecture better than the one following from Tomas-Stein theorem.

9.2. **Wolff's hairbrushes and lower bound** $(n+2)/2$. Bourgain's results were improved by Wolff in [W1]. He got the lower bound $(n+2)/2$ for the Hausdorff dimension of Besicovitch sets, agreeing with 2 in the plane. We shall prove this next. First we formulate another sufficient condition giving lower bound for the dimension of Besicovitch sets. This condition means again boundedness of the Kakeya maximal operator, I shall comment on this a little later.

Theorem 9.3. Let $1 < p < \infty$. Suppose that and $0 < \delta < 1$ for every δ-separated subset $\{e_1, \dots, e_m\}$ of S^{n-1} and for all $a_1, \dots, a_m \in \mathbb{R}^n$,

$$\|\sum_{j=1}^{m} \chi_{T^\delta_{e_j}}(a_j)\|_{L^p(\mathbb{R}^n)} \leq C_\varepsilon \delta^{n/p-n+1-\varepsilon} \tag{9.10}$$

for all $\varepsilon > 0$. Then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $p/(p-1)$.

Proof. The proof proceeds along similar lines as that of Theorem 5.5. Let $B \subset \mathbb{R}^n$ be a Besicovitch set. Then B is compact (though this is not essential as an easy modification of the proof shows) and for every $e \in S^{n-1}$ there is a unit line segment $I_e \subset B$ parallel to e. Let $0 < \alpha < p/(p-1) = p'$ and let $B_j = B(x_j, r_j), j = 1, 2, \dots$, be balls such that $r_j \leq 1/100$ and $B \subset \cup_j B_j$. It suffices to show that $\sum_j r_j^\alpha \gtrsim 1$. For $k = 1, 2, \dots,$ set

$$J_k = \{j : 2^{-k} \leq r_j < 2^{1-k}\},$$

and

$$F_k = \cup_{j \in J_k} B(x_j, 10r_j).$$

By the compactness of B we may assume that there are only finitely balls B_j. Let $\eta > 0$ such that $r_j > \eta$ for all j, and choose a maximal η-separated set

$\{e_1, \dots, e_m\} \subset S^{n-1}$. Then $m \approx \eta^{1-n}$. Set $T_j = T^{\eta}_{e_j}(a_{e_j})$ with a_{e_j} the center of I_{e_j}. Then the balls $10B_i$ cover each T_j , that is,

$$T_j \subset \cup_k F_k.$$

From this we obtain

$$\sum_k \int_{F_k} \sum_j \chi_{T_j} \geq \sum_j \mathcal{L}^n(T_j \cap \cup_k F_k) = \sum_j \mathcal{L}^n(T_j) \approx m\eta^{n-1} \approx 1.$$

It follows that for some k,

$$\int_{F_k} \sum_j \chi_{T_j} \gtrsim k^{-2}.$$

Put $\delta = 2^{-k} > \eta$ for this k. Partition S^{n-1} into disjoint Borel sets $S_j, j = 1, \dots, M \approx \eta^{1-n}$ with $\sigma^{n-1}(S_j) \approx \eta^{n-1}$, and define $\tilde{T}_j = T^{\delta}_{e_j}(a_{e_j}), T(e) = T_j$ and $\tilde{T}(e) = \tilde{T}_j$, when $e \in S_j$. Then by simple geometry,

$$\eta^{1-n}\mathcal{L}^n(T_j \cap F_k) \lesssim \delta^{1-n}\mathcal{L}^n(\tilde{T}_j \cap F_k).$$

Therefore

$$\begin{aligned}
&k^{-2} \lesssim \int_{F_k} \sum_j \chi_{T_j} d\mathcal{L}^n \\
&\approx \eta^{1-n} \sum_j \int_{F_k} \int_{S_j} \chi_{T(e)}(x) d\sigma^{n-1}e d\mathcal{L}^n x = \eta^{1-n} \int \int_{F_k} \chi_{T(e)}(x) d\mathcal{L}^n x d\sigma^{n-1}e \\
&= \int \eta^{1-n}\mathcal{L}^n(T(e) \cap F_k) d\sigma^{n-1}e \lesssim \int \delta^{1-n}\mathcal{L}^n(\tilde{T}(e) \cap F_k) d\sigma^{n-1}e \\
&= \delta^{1-n} \int \int_{F_k} \chi_{\tilde{T}(e)} d\mathcal{L}^n d\sigma^{n-1}e.
\end{aligned}$$

We shall now use the following simple lemma:

Lemma 9.4. If $0 < \delta < 1$ and f is a non-negative Borel function f on S^{n-1} , then there is a δ-separated set $\{u_1, \dots, u_l\} \subset S^{n-1}$ such that $l \approx \delta^{1-n}$ and $\sum_{j=1}^{l} f(u_j) \geq l \int f d\sigma^{n-1}/\sigma^{n-1}(S^{n-1})$.

Proof. Let $\{v_1, \dots, v_l\} \subset S^{n-1}$ be a δ-separated set with $l \approx \delta^{1-n}$ and let θ_n be the Haar measure on on the orthogonal group $O(n)$ with $\theta_n(O(n)) = \sigma^{n-1}(S^{n-1})$. For every $v \in S^{n-1}$ the functional $f \mapsto \int f(g(v)) d\theta_n g$ defines an orthogonally invariant Borel measure on S^{n-1} which, by the uniqueness of such measures, agrees with σ^{n-1}. Hence $\int \sum_{j=1}^{l} f(g(v_j)) d\theta_n g = l \int f d\sigma^{n-1}$, and there is $g \in O(n)$ such that $\sum_{j=1}^{l} f(g(v_j)) \geq l \int f d\sigma^{n-1}/\sigma^{n-1}(S^{n-1})$. So $\{g(v_j) : j = 1, \dots, l\}$ is the desired δ-separated set. □

Applying this with $f(e) = \int_{F_k} \chi_{\tilde{T}(e)} d\mathcal{L}^n$ we find a δ-separated set $\{u_1, \dots, u_l\} \subset S^{n-1}$ and the corresponding tubes $U_j = \tilde{T}(u_j)$ such that $l \approx \delta^{1-n}$ and

$$\int_{F_k} \sum_{j=1}^{l} \chi_{U_j} d\mathcal{L}^n \gtrsim k^{-2}.$$

Hence by Hölder's inequality and (9.10),

$$k^{-2} \lesssim \| \sum_{j=1}^{l} \chi_{U_j} \|_{L^p(\mathbb{R}^n)} \mathcal{L}^n(F_k)^{1/p'} \lesssim C_\varepsilon \delta^{n/p-n+1-\varepsilon} (\# J_k \delta^n)^{1/p'} = C_\varepsilon \delta^{-n/p'+1-\varepsilon} (\# J_k \delta^n)^{1/p'}.$$

Recalling that $\delta = 2^{-k}$, and so $k \approx \log(1/\delta)$, and $\alpha < p'$, we get

$$C_\varepsilon^{p'} \sum_j r_j^\alpha \gtrsim C_\varepsilon^{p'} \# J_k \delta^\alpha \gtrsim \log(1/\delta)^{-2} \delta^{p'(n/p'-1+\varepsilon)-n+\alpha} = \log(1/\delta)^{-2} \delta^{-p'+p'\varepsilon-\alpha} \gtrsim 1$$

as required, when we choose ε small enough. □

Theorem 9.5. Let $0 < \delta < 1$. For every δ-separated subset $\{e_1, \dots, e_m\}$ of S^{n-1} and for all $a_1, \dots, a_m \in \mathbb{R}^n$,

$$\| \sum_{j=1}^{m} \chi_{T_{e_j}^\delta}(a_j) \|_{L^p(\mathbb{R}^n)} \leq C_\varepsilon \delta^{n/p-n+1-\varepsilon} \tag{9.11}$$

for all $\varepsilon > 0$ with $p = (n+2)/n$. In particular, the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $(n+2)/2$.

Proof. The statement about Besicovitch sets follows immediately from Theorem 9.3. Let $T_j = T_{e_j}^\delta(a_j), j = 1, \dots, m$, be as in the theorem. We may assume that $|e_i - e_j| < 1$ for all i and j in order to avoid that far away directions would correspond to nearby tubes. We shall use 'bilinear approach', that is, we write the p'th power of (9.11) as

$$\int (\sum_{j=1}^{m} \chi_{T_j})^p = \int ((\sum_{j=1}^{m} \chi_{T_j})^2)^{p/2} = \int (\sum_{i,j} \chi_{T_i} \chi_{T_j})^{p/2}.$$

Next we split this double sum into parts according to the distance (or angle) between the directions. Let N be the smallest integer such that $2^{-N} < \delta$ and set

$$J_k = \{(i,j) : 2^{-k} < |e_i - e_j| \leq 2^{1-k}\}, k = 1, \dots, N,$$

$$I_0 = \{1, \dots, m\}.$$

Now we have

$$\sum_{i,j} \chi_{T_i} \chi_{T_j} = \sum_{k=1}^{N} \sum_{(i,j) \in J_k} \chi_{T_i} \chi_{T_j} + 2 \sum_{i \in I_0} \chi_{T_i}.$$

Observe that $p/2 < 1$ if $n > 2$, so we cannot use the triangle inequality. But the elementary inequality $(a+b)^q \leq a^q + b^q, a, b > 0, 0 < q \leq 1$, will be enough.

Applying this we obtain

$$\int(\sum_{i=1}^{m}\chi_{T_i})^p \leq \sum_{k=1}^{N}\int(\sum_{(i,j)\in J_k}\chi_{T_i}\chi_{T_j})^{p/2} + 2\int(\sum_{i\in I_0}\chi_{T_i})^{p/2}.$$

Since there are about $\log(1/\delta)$ values of k, the theorem will follow if we can prove for every $k = 1,\dots,N$,

$$\int(\sum_{(i,j)\in J_k}\chi_{T_i}\chi_{T_j})^{p/2} \leq C_\varepsilon\delta^{n-(n-1)p-\varepsilon}, \tag{9.12}$$

and if we have the same estimate for the sum corresponding to I_0. We shall only prove this for J_k, the case of I_0 follows by a slight modification of the argument.

So fix $k \in \{1,\dots,N\}$. Cover S^{n-1} with balls $B(v_l, 2^{-k}), l = 1,\dots,N_k \approx 2^{(n-1)k}$. Then for every pair $(i,j) \in J_k$ $e_i, e_j \in B_l := B(v_l, 2^{1-k})$ for some l. It follows that

$$\int(\sum_{(i,j)\in J_k}\chi_{T_i}\chi_{T_j})^{p/2} \lesssim \sum_{l=1}^{N_k}\int(\sum_{(i,j)\in J_k, e_i,e_j\in B_l}\chi_{T_i}\chi_{T_j})^{p/2}.$$

As $N_k \approx 2^{(n-1)k}$ we are reduced to showing for every l,

$$\int(\sum_{(i,j)\in J_k, e_i,e_j\in B_l}\chi_{T_i}\chi_{T_j})^{p/2} \leq 2^{-(n-1)k}C_\varepsilon\delta^{n-(n-1)p-\varepsilon}. \tag{9.13}$$

Our next step will be to reduce this to the case $k = 1$, that is, $|e_i - e_j| \approx 1$. So suppose we know (9.13) for $k = 1$. Let $k > 1$ and l as above; we may assume that $v_l = (0,\dots,0,1)$. Consider the linear mapping $L, L(x) = (2^{-k-1}x_1\dots,2^{-k-1}x_{n-1},x_n)$. Then $\det L = 2^{-(k+1)(n-1)}$ and $\chi_{T_j}\circ L = \chi_{L^{-1}(T_j)}$. By change of variable,

$$\int(\sum_{(i,j)\in J_k, e_i,e_j\in B_l}\chi_{T_i}\chi_{T_j})^{p/2} = 2^{-(n-1)(k+1)}\int(\sum_{(i,j)\in J_k, e_i,e_j\in B_l}\chi_{L^{-1}(T_i)}\chi_{L^{-1}(T_j)})^{p/2}.$$

The sets $L^{-1}(T_i)$ are contained in $2^{k+1}\delta$-tubes whose directions $L^{-1}(e_i)$ satisfy $1/2 < |L^{-1}(e_i) - L^{-1}(e_j)| \leq 1$ for the pairs (i,j) which appear in the above sum. We can therefore apply our assumption that (9.13) holds for $k = 1$ to get

$$\int(\sum_{(i,j)\in J_k, e_i,e_j\in B_l}\chi_{T_i}\chi_{T_j})^{p/2} \leq C_\varepsilon 2^{-(n-1)k}(2^k\delta)^{n-(n-1)p-\varepsilon} \leq C_\varepsilon 2^{-(n-1)k}\delta^{n-(n-1)p-\varepsilon},$$

as $n-(n-1)p \leq 0$ (recall that $p = (n+2)/2$).

We have now reduced to proving that if $\{e_1,\dots,e_M\}$ is a δ-separated subset of S^{n-1}, $T_i = T^\delta_{e_i}(a_i), i = 1,\dots,M$, for some $a_i \in \mathbb{R}^n$ and $J_0 = \{(i,j) : 1/2 \leq |e_j - e_k| < 1\}$, then

$$\int(\sum_{(i,j)\in J_0}\chi_{T_i}\chi_{T_j})^{p/2} \leq C_\varepsilon\delta^{n-(n-1)p-\varepsilon}. \tag{9.14}$$

Let's make one more reduction: partition S^{n-1} into subsets $S_l, l = 1, \ldots, N(n)$, of diameter less than $1/4$. Then for any pair $(i,j) \in J_0$ there are l and m such that $e_i \in S_l$, $e_j \in S_m$ and $\mathrm{dist}(S_l, S_m) > 1/4$. To prove (9.14) it suffices to consider each such pair (l,m) separately. That is, it suffices to prove that

$$\int((\sum_{i\in I}\chi_{T_i})(\sum_{j\in J}\chi_{T_j}))^{p/2} = \int(\sum_{i\in I, j\in J}\chi_{T_i}\chi_{T_j})^{p/2} \le C_\varepsilon \delta^{n-(n-1)p-\varepsilon} \tag{9.15}$$

where $I, J \subset \{1, \ldots, M\}$ such that $|e_i - e_j| > 1/4$ when $i \in I, j \in J$ and $M \lesssim \delta^{1-n}$.

For $\mu, \nu \in \{1, \ldots, M\}$, set

$$E_{\mu,\nu} = \{x : \mu \le \sum_{i\in I}\chi_{T_i}(x) < 2\mu, \nu \le \sum_{j\in J}\chi_{T_j}(x) < 2\nu\}.$$

Then we have for the left hand side of (9.15)

$$\int((\sum_{i\in I}\chi_{T_i})(\sum_{j\in J}\chi_{T_j}))^{p/2} = \sum_{\mu,\nu}\int_{E_{\mu,\nu}}((\sum_{i\in I}\chi_{T_i})(\sum_{j\in J}\chi_{T_j}))^{p/2} \le \sum_{\mu,\nu}(4\mu\nu)^{p/2}\mathcal{L}^n(E_{\mu,\nu}),$$

where the summation is over the dyadic integers μ and ν of the form $2^l \le M, l \in \mathbb{N}$. There are only $\lesssim \log(1/\delta)^2$ pairs of them. Thus we can find such a pair (μ,ν) for which

$$\int_{E_{\mu,\nu}}((\sum_{j\in I}\chi_{T_i})(\sum_{j\in J}\chi_{T_j}))^{p/2} \lesssim C_\varepsilon \delta^{-\varepsilon}(\mu\nu)^{p/2}\mathcal{L}^n(E_{\mu,\nu}).$$

Taking also into account that $p = (n+2)/n$, (9.15) is now reduced to

$$(\mu\nu)^{(n+2)/(2n)}\mathcal{L}^n(E_{\mu,\nu}) \lesssim C_\varepsilon \delta^{(2-n)/n-\varepsilon}. \tag{9.16}$$

Keeping the pair (μ,ν) which we found fixed, we define for dyadic rationals κ and λ of the form $2^{-l}, l \in \mathbb{N}$,

$$I_\kappa = \{i \in I : (\kappa/2)\mathcal{L}^n(T_i) \le \mathcal{L}^n(T_i \cap E_{\mu,\nu}) < \kappa\mathcal{L}^n(T_i)\},$$

$$J_\lambda = \{j \in J : (\lambda/2)\mathcal{L}^n(T_j) \le \mathcal{L}^n(T_j \cap E_{\mu,\nu}) < \lambda\mathcal{L}^n(T_j)\}.$$

By the definition of $E_{\mu,\nu}$,

$$\int_{E_{\mu,\nu}}\sum_{i\in I}\sum_{j\in J}\chi_{T_i}\chi_{T_j} \approx \mu\nu\mathcal{L}^n(E_{\mu,\nu}).$$

We can write this as

$$\sum_{\kappa,\lambda}\int\sum_{i\in I_\kappa}\sum_{j\in J_\lambda}\chi_{T_i}\chi_{T_j} \approx \mu\nu\mathcal{L}^n(E_{\mu,\nu}),$$

where the summation in κ and λ is over dyadic rationals as above. We can restrict to κ and λ at least δ^n, since, for example,

$$\sum_{\kappa\leq\delta^n}\int_{E_{\mu,\nu}}\sum_{i\in I_\kappa}\sum_{j\in J_\lambda}\chi_{T_i}\chi_{T_j}\leq\sum_{l=1}^{\infty}\sum_{2^{-l}\delta^n\leq\kappa\leq 2^{1-l}\delta^n}\#J\int_{E_{\mu,\nu}}\sum_{i\in I_\kappa}\chi_{T_i}$$
$$\lesssim\delta^{1-n}\sum_{l=1}^{\infty}\sum_{i\in I_\kappa}2^{1-l}\delta^n\mathcal{L}^n(T_i)\lesssim\delta\leq\delta^{(2-n)/n}.$$

Thus we have again only $\approx\log(1/\delta)$ values to consider and we find and fix κ and λ for which

$$\mu\nu\mathcal{L}^n(E_{\mu,\nu})\leq C_\varepsilon\delta^{-\varepsilon}\int_{E_{\mu,\nu}}\sum_{i\in I_\kappa}\sum_{j\in J_\lambda}\chi_{T_i}\chi_{T_j}. \tag{9.17}$$

Then by the definition of $E_{\mu,\nu}$,

$$\mu\nu\mathcal{L}^n(E_{\mu,\nu})\leq C_\varepsilon\delta^{-\varepsilon}\nu\int_{E_{\mu,\nu}}\sum_{i\in I_\kappa}\chi_{T_i}=C_\varepsilon\delta^{-\varepsilon}\nu\sum_{i\in I_\kappa}\mathcal{L}^n(E_{\mu,\nu}\cap T_i)$$
$$\leq C_\varepsilon\delta^{-\varepsilon}\nu\kappa\sum_{i\in I_\kappa}\mathcal{L}^n(T_i)\lesssim C_\varepsilon\delta^{-\varepsilon}\nu\kappa\#I_\kappa\delta^{n-1}\lesssim C_\varepsilon\delta^{-\varepsilon}\nu\kappa,$$

because $\#I_\kappa\lesssim\delta^{1-n}$. Thus,

$$\mu\mathcal{L}^n(E_{\mu,\nu})\lesssim C_\varepsilon\delta^{-\varepsilon}\kappa. \tag{9.18}$$

We deduce from (9.17),

$$\mu\nu\mathcal{L}^n(E_{\mu,\nu})\lesssim C_\varepsilon\delta^{-\varepsilon}\sum_{j\in J_\lambda}\int_{T_j}\sum_{i\in I_\kappa}\chi_{T_i}\chi_{T_j}.$$

Again, $\#J_\lambda\lesssim\delta^{1-n}$, whence we find and fix $j\in J_\lambda$ such that

$$\delta^{n-1}\mu\nu\mathcal{L}^n(E_{\mu,\nu})\lesssim C_\varepsilon\delta^{-\varepsilon}\int_{T_j}\sum_{i\in I_\kappa}\chi_{T_i}\chi_{T_j}=C_\varepsilon\delta^{-\varepsilon}\sum_{i\in I_\kappa}\mathcal{L}^n(T_i\cap T_j).$$

Since above the directions of T_i and T_j are separated by $1/2$, it follows that $\mathcal{L}^n(T_i\cap T_j)\lesssim\delta^n$, and we conclude

$$\delta^{n-1}\mu\nu\mathcal{L}^n(E_{\mu,\nu})\leq C_\varepsilon\delta^{-\varepsilon}\delta^n\#\{i\in I_\kappa:T_i\cap T_j\neq\varnothing\}.$$

Now we have found Wolff's hairbrush: tubes T_i, on the number of which we have control, intersecting a fixed tube T_j. Next we shall make use of this in a somewhat similar manner as we used Bourgain's bushes in the proof of Theorem 9.2.

So now we have fixed μ,ν,κ,λ and $j\in J_\lambda$. Denote

$$\tilde{I}=\{i\in I_\kappa:T_i\cap T_j\neq\varnothing\}$$

so that

$$\delta^{-1}\mu\nu\mathcal{L}^n(E_{\mu,\nu})\leq C_\varepsilon\delta^{-\varepsilon}\#\tilde{I}. \tag{9.19}$$

Then for $i \in \tilde{I}$, $\mathcal{L}^n(T_i \cap E_{\mu,\nu}) \geq (\kappa/2)\mathcal{L}^(T_i)$. By simple geometry there is a positive constant b depending only on n such that when we set

$$U = \{x \in \mathbb{R}^n : \operatorname{dist}(x, T_j) > b\kappa\},$$

we have for $i \in \tilde{I}$, $\mathcal{L}^n(T_i \setminus U) < (\kappa/4)\mathcal{L}^n(T_i)$; recall that the directions e_i and e_j of T_i and T_j satisfy $|e_i - e_j| \geq 1/2$. Therefore

$$\mathcal{L}^n(T_i \cap E_{\mu,\nu} \cap U) \geq (\kappa/4)\mathcal{L}^n(T_i).$$

Summing over i gives

$$\int_{E_{\mu,\nu}} \sum_{i \in \tilde{I}} \chi_{T_i \cap U} \gtrsim \kappa\delta^{n-1}\#\tilde{I}.$$

By Schwartz's inequality,

$$\kappa\delta^{n-1}\#\tilde{I} \lesssim \|\sum_{i \in \tilde{I}} \chi_{T_i \cap U}\|_2 \mathcal{L}^n(E_{\mu,\nu})^{1/2}.$$

We shall prove that

$$\|\sum_{i \in \tilde{I}} \chi_{T_i \cap U}\|_2 \lesssim (\kappa^{2-n}\delta^{n-1}\#\tilde{I})^{1/2}. \tag{9.20}$$

Let us first see how we can complete the proof of the theorem from this.

Combining (9.20) with the previous inequality, we obtain

$$\mathcal{L}^n(E_{\mu,\nu}) \gtrsim \kappa^n\delta^{n-1}\#\tilde{I}.$$

Bringing in (9.19) we get

$$\kappa^n\delta^{n-2}\mu\nu \leq C_\varepsilon\delta^{-e}.$$

Recalling also (9.16) this gives

$$\mu^{n+1}\nu\mathcal{L}^n(E_{\mu,\nu})^n \lesssim \delta^{2-n}.$$

Interchanging μ and ν,

$$\mu\nu^{n+1}\mathcal{L}^n(E_{\mu,\nu})^n \lesssim \delta^{2-n}.$$

Thus

$$\mathcal{L}^n(E_{\mu,\nu})^n \lesssim \sqrt{(\mu^{-n-1}\nu\delta^{2-n})(\mu\nu^{-n-1}\delta^{2-n})} = (\mu\nu)^{-(n+2)/2}\delta^{2-n},$$

which is the desired inequality (9.16).

We have still left to prove (9.20). The square of the left hand side of it is

$$\int (\sum_{i \in \tilde{I}} \chi_{T_i \cap U})^2 = \sum_{i,i' \in \tilde{I}} \mathcal{L}^n(T_i \cap T_{i'} \cap U) \lesssim \kappa^{2-n}\delta^{n-1}\#\tilde{I},$$

provided we can show for every $i' \in \tilde{I}$,

$$\sum_{i \in \tilde{I}} \mathcal{L}^n(T_i \cap T_{i'} \cap U) \lesssim \kappa^{2-n}\delta^{n-1}. \tag{9.21}$$

Obviously it suffices to sum over $i \neq j$. We split this into the sums over

$$\tilde{I}_k = \{i \in \tilde{I} : 2^{-k} \leq |e_i - e_j| < 2^{1-k}\}, T_i \cap T_{i'} \cap U \neq \varnothing\}, k = 1, \ldots, N \approx \log(1/\delta) :$$

$$\sum_{i\in\tilde{I}}\mathcal{L}^n(T_i\cap T_j\cap U)=\sum_{k=1}^{N}\sum_{i\in\tilde{I}_k}\mathcal{L}^n(T_i\cap T_{i'}\cap U)\lesssim\sum_{k=1}^{N}\#\tilde{I}_k 2^k\delta^n,$$

since, as before, by simple geometry, $\mathcal{L}^n(T_i\cap T_{i'}\cap U)\lesssim 2^k\delta^n$ for $i\in\tilde{I}_k$. Once more we use the fact that there are no more than about $\log(1/\delta)$ terms in this sum to reduce (9.21) to

$$\#I_k 2^k\delta^n\leq\kappa^{2-n}\delta^{n-1}. \tag{9.22}$$

To see where this geometric fact follows from let us recall the situation. We have fixed the two tubes T_j and $T_{i'}$ which intersect at an angle ≈ 1. For $i\in\tilde{I}_k$, the tube T_i intersects both of these and $T_{i'}$ it intersects in U, that is, outside a $b\kappa$-neighborhood of T_j. This means that these three tubes lie close to a two-dimensional plane, in fact roughly at a distance δ/κ from a plane. After this observation we leave it as an exercise to deduce (9.22) and finish the proof of the theorem. □

9.3. **Bourgain's arithmetic method and lower bound $cn+1-c$.** Bourgain proved in [B1] that the Hausdorff dimension of all Besicovitch sets in $\mathbb{R}^n$ is at least $\frac{13}{25}n+\frac{12}{25}$. This estimate is better than Wolff's $\frac{n+2}{2}$ only if $n>26$, but in high dimensions it is an improvement. Perhaps more interesting than the estimate itself is the arithmetic method Bourgain introduced. He also used it to get L^p-estimates for the Kakeya maximal operator.

Bourgain's proof is fairly complicated. Here I only present a special case of the result, a similar estimate for Minkowski (or box counting) dimension. The presentation is based Tao's lecture notes from his UCLA web page.

Definition 9.6. The (lower) Minkowski dimension of a bounded set $A\subset\mathbb{R}^n$ is

$$\dim_M A=\inf\{s>0:\liminf_{\delta\to 0}\delta^{s-n}\mathcal{L}^n(N_\delta(A))=0\},$$

where $N_\delta(A)=\{x:\operatorname{dist}(x,A)<\delta\}$ is the open δ-neighborhood of A.

It is easy to easy show (or see [M], for example) that the Minkowski dimension is always at least as big as the Hausdorff dimension. Hence the result below is considerably weaker than Bourgain's result.

Theorem 9.7. For any Besicovitch set B in $\mathbb{R}^n$, $\dim_M B\geq c_0 n+1-c_0$ with $c_0=\frac{153}{305}>1/2$.

Note that $c_0>1/2$ so that this bound is also greater than $(n+2)/2$ for large n. It is clear that small modifications in the proof would give a bigger c_0. We shall prove the theorem for slightly modified Besicovitch sets, but it can readily be reduced to these. Namely, we assume that $B\subset[0,1]^n$ and for every $v\in[0,1]^{n-1}$ there is $x\in[0,1]^{n-1}$ such that B contains the line segment

$$I(x,v):=\{(x,0)+t(v,1):0\leq t\leq 1\}.$$

We make the counterassumption that $\dim_M B < cn + 1 - c$ for some $c < c_0$ and try to achieve a contradiction. By the definition of the Minkowski dimension

$$\mathcal{L}^n(N_{2\delta}(B)) \leq \delta^{(1-c)(n-1)}$$

for some arbitrarily small $\delta > 0$, which we now fix for a moment. For any $A \subset \mathbb{R}^n$ let

$$A(t) = A \cap (\mathbb{R}^{n-1} \times \{t\})$$

be the horizontal slice of A at the level t. By Fubini's theorem

$$\int_0^1 \mathcal{L}^{n-1}(N_{2\delta}(B)(t)) \leq \delta^{(1-c)(n-1)},$$

so Chebyshev's inequality gives,

$$\mathcal{L}^1(\{t \in [0,1] : \mathcal{L}^{n-1}(N_{2\delta}(B)(t)) > 100\delta^{(1-c)(n-1)}\}) < 1/100.$$

Setting

$$A = \{t \in [0,1] : \mathcal{L}^{n-1}(N_{2\delta}(B)(t)) \leq 100\delta^{(1-c)(n-1)}\}$$

we have $\mathcal{L}^1(A) > 99/100$. From this it follows (as an easy exercise) that there are $s, s+d, s+2d \in A$ with $d > 1/10$. (In fact, any measurable subset of $\mathbb{R}$ with positive measure contains an arithmetic progression of length 3 due to a theorem of Roth, but this is not so easy anymore.) We can assume that $s = 0$ and $d = 1/2$ so that our numbers are now $0, 1/2$ and 1.

For $t \in [0,1]$ set

$$B[t] = \{i \in \delta\mathbb{Z}^{n-1} : (i,t) \in N_\delta(B)\}.$$

Then the balls $B((i,t), \delta/3), i \in B[t]$, are disjoint and contained in $N_{2\delta}(B)$. Combining this with the fact that $0, 1/2, 1 \in A$ we obtain by a simple measure comparision

$$\sharp B[0],\ \sharp B[1/2],\ \sharp B[1] \lesssim_n \delta^{c(1-n)}. \tag{9.23}$$

Define for $u, v \in \mathbb{R}^{n-1}$ the δ-tubes $T_\delta^v(u) = \{y \in \mathbb{R}^n : \operatorname{dist}(y, I(u,v)) < \delta\}$ modified to our situation, and

$$G = \{(x,y) \in B[0] \times B[1] : (x,0), (y,1) \in T_\delta^v(u) \subset N_\delta(B) \text{ for some } u, v \in [0,1]^{n-1}\}.$$

Then

$$\sharp\{x + y \in G : (x,y) \in G\} \lesssim_n \delta^{c(1-n)}$$

and

$$\sharp\{x - y \in G : (x,y) \in G\} \gtrsim_n \delta^{1-n}.$$

To check the first of these inequalities observe that for $(x,y) \in G$, $((x+y)/2, 1/2)$ belongs to the same tube as $(x,0)$ and $(x,1)$, so it belongs to $N_\delta(B)$. Since it also belongs to $\frac{1}{2}\delta\mathbb{Z}^{n-1}$, the cardinality of $\{x + y \in G : (x,y) \in G\}$ is dominated by the cardinality of $B[1/2]$, and the first inequality follows. The second inequality is a consequence of the Besicovitch property of B: there are roughly δ^{1-n} δ-tubes with δ-separated directions contained in $N_\delta(B)$, each of these contains points $(x,0)$ and $(y,1)$ for some $(x,y) \in G$ and for different tubes the differences $x - y$, essentially directions of these tubes, are different.

So the sum set of G is small and its difference set is large. We shall derive a contradiction from this using the following proposition:

Proposition 9.8. Let $\varepsilon_0 = \frac{1}{153}$. There exists a positive number N_0 with the following property. Suppose that A and B are finite subsets of $\lambda\mathbb{Z}^m$ for some $m \in \mathbb{N}$ and $\lambda > 0$, $\#A \leq N$ and $\#B \leq N$. Suppose $G \subset A \times B$ and

$$\sharp\{x + y \in G : (x, y) \in G\} \leq N. \tag{9.24}$$

Then for $N > N_0$,

$$\sharp\{x - y \in G : (x, y) \in G\} \leq N^{2-\varepsilon_0}. \tag{9.25}$$

This is a purely combinatorial proposition and, as will be clear from the proof, it holds for any free Abelian group in place of $\lambda\mathbb{Z}^m$. Theorem 9.7 follows applying the proposition to what we did before with $N = \delta^{c(1-n)}$ if δ is sufficiently small.

Observe that the proposition trivial for $\varepsilon_0 = 0$. The application of this gives anyway $\dim_M B \geq (n+1)/2$, which is not completely trivial but much less than we already know.

Now we begin the proof of Proposition 9.8 by assuming that it is false. Then we have $A, B \subset \lambda\mathbb{Z}^m$ and large N with

$$\sharp A, \sharp B \leq N, \tag{9.26}$$

and $G \subset A \times B$ satisfying (9.24) and

$$\#\{x - y \in G : (x, y) \in G\} > N^{2-\varepsilon_0}, \tag{9.27}$$

which of course yield

$$N^{2-\varepsilon_0} \lesssim \#G \leq N^2. \tag{9.28}$$

Here and later in this proof the implicit constant in $\lesssim$ is absolute. Above we could take it 1, we modify G below and want (9.28) still to hold. We can assume that

$$(x, y) \mapsto x - y \text{ is one-to-one on } G \tag{9.29}$$

simply by replacing G with a subset obtained by choosing one pair for every difference and removing the rest from G. Denote

$$G^b = \{a \in A : (a, b) \in G\} \text{ for } b \in B.$$

We may assume that

$$\#G^b \geq N^{1-2\varepsilon_0} \text{ for } b \in B, \tag{9.30}$$

because removing those $G^b \times \{b\}$'s from G for which $\#G^b < N^{1-2\varepsilon_0}$, say $b \in B_0$, reduces the cardinality of G only by $\sum_{b \in B_0} \#G^b < N^{2-2\varepsilon_0}$ and (9.28) remains valid provided N is large enough.

The sum set being small means that many pairs have the same sum, and we would like to have this property also for the differences. The key to this is the trivial identity

$$a + b = a' + b' \iff a - b' = a' - b,$$

which we shall employ in the proof of the following lemma.

Lemma 9.9. Let $A, B \subset \lambda\mathbb{Z}^m, G \subset A \times B$ and $N \in \mathbb{N}$ such that $\#A \leq N, \#B \leq N$, (9.24) and (9.28) hold. Then there exists $I \subset A \times B$ of cardinality

$$\#I \gtrsim N^{2-5\varepsilon_0} \tag{9.31}$$

such that for all $(a, b) \in I$,

$$\#\{(a', b') \in A \times B : a - b = a' - b'\} \geq N^{1-3\varepsilon_0}. \tag{9.32}$$

Proof. Define the sum counting function s on $C := \{a + b : (a, b) \in G\}$ by

$$s(x) = \#\{(a, b) \in G : a + b = x\}.$$

Then by (9.28), Schwartz's inequality and (9.24) we have

$$N^{2-\varepsilon_0} \lesssim \#G = \sum_{x \in C} s(x) \leq (\#C)^{1/2} (\sum_{x \in C} s(x)^2)^{1/2} \leq N^{1/2} (\sum_{x \in C} s(x)^2)^{1/2},$$

whence

$$\sum_{x \in C} s(x)^2 \gtrsim N^{3-2\varepsilon_0}.$$

This implies

$$\#\{(a, b, a', b') \in A \times B \times A \times B : a + b = a' + b'\} \gtrsim N^{3-2\varepsilon_0},$$

or equivalently,

$$\#\{(a, b, a', b') \in A \times B \times A \times B : a - b' = a' - b\} \gtrsim N^{3-2\varepsilon_0}.$$

Define the difference counting function d on $\{a - b : (a, b) \in A \times B\}$ by

$$d(x) = \#\{(a, b) \in A \times B : a - b = x\}.$$

Then the last inequality turns into

$$\sum_x d(x)^2 \gtrsim N^{3-2\varepsilon_0}. \tag{9.33}$$

Since $\#(A \times B) \leq N^2$, we have

$$\sum_x d(x) \leq N^2,$$

so

$$\sum_{d(x) \leq N^{1-3\varepsilon_0}} d(x)^2 \leq N^{3-3\varepsilon_0},$$

and further using also (9.33),

$$\sum_{d(x) > N^{1-3\varepsilon_0}} d(x)^2 \gtrsim N^{3-2\varepsilon_0}.$$

Combining this with the obvious fact that $d(x) \leq N$ for every x, we get

$$\#\{x : d(x) > N^{1-3\varepsilon_0}\} \gtrsim N^{1-2\varepsilon_0}.$$

Letting I be the set of pairs (a, b) such that $d(a - b) > N^{1-3\varepsilon_0}$, (9.32) follows from the definition of I. Also (9.31) follows since every difference $x = a - b, (a, b) \in I$, is realized by at least $N^{1-3\varepsilon_0}$ pairs $(a', b') \in A \times B$, which all are in I by the definition of I, and there are at least $N^{1-2\varepsilon_0}$ such differences. $\square$

Next we shall make use of I and the identity

$$a - b = (a - b') - (a' - b') + (a' - b).$$

If we could find about N^2 (up to suitable $N^{-\varepsilon}$) pairs (a, b) having this representation for about N^2 pairs (a', b') in such a way that the pairs $(a, b'), (a', b')$ and (a', b) would belong to I, we would be done. Namely, then we could count by the previous lemma about N^7 (up to suitable $N^{-\varepsilon}$) different expressions $(a_1 - b_1) - (a_2 - b_2) + (a_3 - b_3)$. But there are only at most N^6 six-tuples $(a_1, b_1, a_2, b_2, a_3, b_3)$ altogether which would lead to a contradiction. Now we proceed to do this more precisely.

Let us say that $b \in B$ and $b' \in B$ communicate, written as $b \sim b'$, if

$$\#\{a \in A : (a, b), (a, b') \in I\} \geq N^{1-55\varepsilon_0}.$$

Lemma 9.10. There exists $B' \subset B$ such that $\#B' \gtrsim N^{1-5\varepsilon_0}$ and

$$\#\{(b, b') \in B' \times B' : b \not\sim b'\} \lesssim N^{2-44\varepsilon_0}. \tag{9.34}$$

Proof. We shall find B' as one of the sections

$$I_a := \{b \in B : (a, b) \in I\}.$$

By the definition of the communicavity,

$$\sum_{a \in A} \#\{(b, b') \in I_a \times I_a : b \not\sim b'\} = \sum_{b \not\sim b'} \#\{a \in A : (a, b), (a, b') \in I\}$$
$$\leq \sum_{b \not\sim b'} N^{1-55\varepsilon_0} \leq N^{3-55\varepsilon_0}.$$

On the other hand, we have by Schwartz's inequality and (9.31),

$$\sum_{a \in A} (\#I_a)^2 \geq (\#A)^{-1} (\sum_{a \in A} \#I_a)^2 = (\#A)^{-1} (\#I)^2 \gtrsim N^{3-10\varepsilon_0}.$$

Consequently,

$$\sum_{a \in A} (\#I_a)^2 - N^{44\varepsilon_0} \sum_{a \in A} \#\{(b, b') \in I_a \times I_a : b \not\sim b'\} \gtrsim N^{3-10\varepsilon_0}.$$

Since the cardinality of A is at most N, there must exist $a \in A$ such that

$$(\#I_a)^2 - N^{44\varepsilon_0} \#\{(b, b') \in I_a \times I_a : b \not\sim b'\} \gtrsim N^{2-10\varepsilon_0}.$$

Then $\#I_a \gtrsim N^{1-5\varepsilon_0}$ and $\#\{(b, b') \in I_a \times I_a : b \not\sim b'\} \leq N^{2-44\varepsilon_0}$, so that letting $B' = I_a$, the lemma follows. $\square$

We have by (9.30)

$$\#(G \cap (A \times B')) = \sum_{b \in B'} \#G^b \gtrsim N^{2-7\varepsilon_0}.$$

Let us define now

$$G_a = \{b \in B' : (a, b) \in G\}$$

and

$$A' = \{a \in A : \#G_a \geq N^{1-8\varepsilon_0}\}.$$

Then arguing as for (9.30),

$$\#A' \gtrsim N^{1-7\varepsilon_0} \text{ and } \#(G \cap (A' \times B')) \gtrsim N^{2-7\varepsilon_0}.$$

We apply now Lemma 9.9 to $A', B', G \cap (A' \times B')$ and $7\varepsilon_0$ in place of A, B, G and ε_0 to find $I' \subset A' \times B'$ of cardinality $\#I' \gtrsim N^{2-35\varepsilon_0}$ such that for all $(a, b) \in I'$ the difference $a - b$ can be written roughly at least in $N^{1-21\varepsilon_0}$ different ways as $a' - b'$ with $a' \in A', b' \in B'$.

Now we do some counting. First

$$\#\{(a, b, b') \in A' \times B' \times B' : (a, b) \in G, (a, b') \in I'\} \gtrsim N^{3-43\varepsilon_0},$$

because there are (up to a constant multiplication) at least $N^{2-35\varepsilon_0}$ choices of (a, b') and then at least $N^{1-8\varepsilon_0}$ ways to choose $b \in G_a$. On the other hand,

$$\#\{(a, b, b') \in A' \times B' \times B' : (a, b) \in G, (a, b') \in I', b \not\sim b'\} \lesssim N^{3-44\varepsilon_0},$$

because there are at most $N^{2-44\varepsilon_0}$ choices of (b, b') and then at most N ways to choose a. Combining these we find that

$$\#\{(a, b, b') \in A' \times B' \times B' : (a, b) \in G, (a, b') \in I', b \sim b'\} \gtrsim N^{3-43\varepsilon_0}.$$

We can write this as

$$\sum_{(a,b) \in G \cap (A' \times B')} \#\{b' \in B' : (a, b') \in I', b \sim b'\} \gtrsim N^{3-43\varepsilon_0}.$$

There are at most N^2 pairs in the sum, and each summand is at most N. So there must exist at least $N^{2-44\varepsilon_0}$ pairs $(a, b) \in G \cap (A' \times B')$ such that

$$\#\{b' \in B' : (a, b') \in I', b \sim b'\} \gtrsim N^{1-44\varepsilon_0}.$$

Therefore we have by the definition of the communicavity

$$\#\{(a', b') \in A' \times B' : (a, b') \in I', (a', b), (a', b') \in I\} \gtrsim N^{2-99\varepsilon_0}$$

for these pairs (a, b). Writing $a - b = (a - b') - (a' - b') + (a' - b)$, it follows from Lemma 9.9 that for at least $N^{2-44\varepsilon_0}$ pairs (a, b),

$$\#\{(a_1, b_1, a_2, b_2, a_3, b_3) : a - b = (a_1 - b_1) - (a_2 - b_2) + (a_3 - b_3)\} \gtrsim N^{5-108\varepsilon_0}.$$

For this we would need at least $N^{7-152\varepsilon_0} = N^{7-152/153} > N^6$ different six-tuples $(a_1, b_1, a_2, b_2, a_3, b_3)$, but there are no more than N^6 of them and we have achieved a contradiction, which proves proves Proposition 9.8, and so also Theorem 9.7.

10. BILINEAR RESTRICTION

10.1. **Bilinear vs. linear restriction.** Earlier we studied the restriction inequalities (in the dual form)

(10.1) $$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(S^{n-1})} \text{ for } f \in L^p(S^{n-1}).$$

Recall that by $\hat{f}$ we mean here the Fourier transform of the measure $f\sigma^{n-1}$. By Hölder's inequality we can write this in an equivalent form

(10.2) $$\|\hat{f}_1\hat{f}_2\|_{L^{q/2}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^p(S^{n-1})}\|f_2\|_{L^p(S^{n-1})} \text{ for } f_1, f_2 \in L^p(S^{n-1}).$$

As such there is not much gain but if f_1 and f_2 are supported in different parts of the sphere, we can get something better. Let us look briefly at the case $p = 2, q = 4$. Suppose that the distance between the supports of f_1 and f_2 is greater than some absolute constant $c_0 > 0$. By Plancherel's theorem the inequality

$$\|\hat{f}_1\hat{f}_2\|_{L^2(\mathbb{R}^n)} \lesssim \|f_1\|_{L^2(S^{n-1})}\|f_2\|_{L^2(S^{n-1})}$$

reduces to the non-Fourier statement

$$\|(f_1\sigma^{n-1}) * (f_2\sigma^{n-1})\|_{L^2(\mathbb{R}^n)} \lesssim \|f_1\|_{L^2(S^{n-1})}\|f_2\|_{L^2(S^{n-1})},$$

which is not very difficult to prove, although not trivial either. On the other hand, the corresponding linear inequality

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^2(S^{n-1})}$$

is the Tomas-Stein theorem and it holds if and only if $q \geq (2n+2)/(n-1)$, recall Theorem 4.4.

The bilinear restriction problem on the sphere asks for what exponents p and q the inequality (10.2) holds for $f_j \in L^p(S^{n-1}), j = 1, 2$, or for $f_j \in \mathcal{S}$, if $\operatorname{spt} f_j \subset S_j$ and $S_j \subset S^{n-1}$ with $\operatorname{dist}(S_1, S_2) \gtrsim 1$. More generally, S_1 and S_2 can be some other type of surfaces (pieces of paraboloids, cones, etc.). The essential condition required is usually that they are transversal, that is, their normals point to separated directions.

The point in bilinear estimates is not only, nor mainly, in getting new types of inequalities, but it is in their applications. In particular, they can be used to improve the linear estimates, and that is what we are going to discuss here. One way (and equivalent to others we have met) to state the restriction conjecture is

Conjecture 10.1.

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(S^{n-1})} \text{ for } f \in L^p(S^{n-1}), p' \leq \frac{n-1}{n+1}q, q > 2n/(n-1).$$

By the Tomas-Stein theorem this is valid for $p = 2, q = (2n+2)/(n-1)$, and whence by interpolation for $q \geq (2n+2)/(n-1)$. The Kakeya methods developed by Bourgain and Wolff give some improvements for this, but still better results can be obtained via bilinear restriction. This is based on two facts: a general result of Tao, Vargas and Vega from [TVV] of the type 'bilinear restriction estimates imply linear ones' and a bilinear restriction theorem of Tao from [Tao1]. The latter is the following:

Theorem 10.2. Let $S_j \subset S^{n-1}, j = 1, 2$, with $\operatorname{dist}(S_1, S_2) \gtrsim 1$ and $f_j \in \mathcal{S}$ with $\operatorname{spt} f_j \subset S_j$. Then

$$\|\hat{f}_1 \hat{f}_2\|_{L^q(\mathbb{R}^n)} \lesssim \|f_1\|_{L^2(S_1)} \|f_2\|_{L^2(S_2)} \text{ for } q > (n+2)/n.$$

The lower bound $(n+2)/2$ is the best possible by rather easy examples. In fact, Tao proved his results for paraboloids, but as he says in the paper, the method works for more general surfaces including spheres. The class of surfaces was further extended by Lee in [Lee]. The proof is long and complicated, we shall discuss some parts of it a little later. First we look at the result of Tao, Vargas and Vega:

Theorem 10.3. Let $1 < p, q < \infty, q > 2n/(n-1)$ and $p' < \frac{n-1}{n+1}q$. If the estimate

$$\|\hat{f}_1 \hat{f}_2\|_{L^{q/2}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^p(S_1)} \|f_2\|_{L^p(S_2)}, \tag{10.3}$$

holds for transversal surfaces S_1 and S_2, then also the estimate

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(S^{n-1})}$$

holds.

Concerning the class of surfaces involved, the formulation here is rather unprecise, see [TVV] for the precise one. In the proof one needs the bilinear estimates for a class of surfaces which are obtained from spherical caps via affine transformations. The result in [TVV] also includes the case $p' = \frac{n-1}{n+1}q$.

Combining the last two theorems (and remembering the above remark on [TVV]), we obtain

Theorem 10.4. The restriction conjecture holds for $q > 2(n+2)/n$:

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(S^{n-1})} \text{ for } f \in L^p(S^{n-1}), p' \leq \frac{n-1}{n+1}q, q > 2(n+2)/n.$$

We now sketch the proof of Theorem 10.3. We only consider the case where $q \leq 4$. This is actually enough by the Tomas-Stein theorem and the fact that the restriction conjecture is valid in the plane.

Suppose f has support in a part of S^{n-1} which has a parametrization $(v, \varphi(v)), v \in Q$, where Q is a cube in $\mathbb{R}^{n-1}$ and $\varphi > 0$. Then we can write the Fourier transform of f (forgetting the Jacobian term)

$$\hat{f}(x,t) = \int_Q e^{-2\pi i(x\cdot v + t\varphi(v))} f(v, \varphi(v)) dv, \ (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Next we write

$$\|\hat{f}\|^2_{L^q(\mathbb{R}^n)} = \|(\hat{f})^2\|_{L^{q/2}(\mathbb{R}^n)},$$

and

$$(\hat{f})^2(x,t) = \int_Q \int_Q e^{-2\pi i(x\cdot v + t\varphi(v))} f(v, \varphi(v)) e^{-2\pi i(x\cdot w + t\varphi(w))} f(w, \varphi(w)) dv dw.$$

We introduce a Whitney decomposition of $Q \times Q \setminus \Delta, \Delta = \{(v,w) : v = w\}$, into disjoint cubes $I \times J \in \mathcal{Q}_k, k = 1, 2, \dots$, where I and J are dyadic subcubes of Q such that $\operatorname{diam}(I) \approx \operatorname{diam}(J) \approx \operatorname{dist}(I,J) \approx 2^{-k}$ when $I \times J \in \mathcal{Q}_k$. Let $f_I(v,\varphi(v)) = f(v,\varphi(v))\chi_I(v)$. Then we have

$$\|\hat{f}\|^2_{L^q(\mathbb{R}^n)} = \|\sum_k \sum_{I\times J\in\mathcal{Q}_k} \widehat{f_I}\widehat{f_J}\|_{L^{q/2}(\mathbb{R}^n)} \leq \sum_k \|\sum_{I\times J\in\mathcal{Q}_k} \widehat{f_I}\widehat{f_J}\|_{L^{q/2}(\mathbb{R}^n)}. \tag{10.4}$$

The inverse transform of $\widehat{f_I}\widehat{f_J}$ is the convolution $f_I * f_J$. We have for its support

$$\operatorname{spt} f_I * f_J \subset S(I,J) := \{(v,t) : v \in I+J, 0 \leq t \leq 2\}.$$

Here $I+J$ lies in a $C2^{-k}$-neighbourhood of $I+I$ for some absolute constant C, so the sets $S(I,J), I \times J \in \mathcal{Q}_k$, have bounded overlap for each fixed k. Choose smooth compactly supported functions $\psi(I,J) \leq 1$ such that $\psi(I,J) = 1$ on $S(I,J)$, $\|\mathcal{F}(\psi(I,J))\|_1 \approx 1$, and

$$\sum_{I\times J\in\mathcal{Q}_k} \chi_{\operatorname{spt}\psi(I,J)} \lesssim 1, \tag{10.5}$$

and define the operators $T_{I,J}$ by

$$T_{I,J}g = \mathcal{F}(\psi(I,J)\check{g}).$$

Using (10.5) Plancherel's theorem gives the L^2-estimate for arbitrary L^2-functions $g_{I,J}$,

$$\|\sum_{I\times J\in\mathcal{Q}_k} T_{I,J}g_{I,J}\|^2_{L^2(\mathbb{R}^n)} \lesssim \sum_{I\times J\in\mathcal{Q}_k} \|g_{I,J}\|^2_{L^2(\mathbb{R}^n)}.$$

The L^1-estimate

$$\|\sum_{I\times J\in\mathcal{Q}_k} T_{I,J}g_{I,J}\|_{L^1(\mathbb{R}^n)} \lesssim \sum_{I\times J\in\mathcal{Q}_k} \|g_{I,J}\|_{L^1(\mathbb{R}^n)}$$

for arbitrary L^1-functions $g_{I,J}$ follows by $\|T_{I,J}g_{I,J}\|_1 \lesssim \|g_{I,J}\|_1$ and triangle inequality. These two inequalities tell us that the operator $T_k, T_k(g_{I,J}) = \sum_{I\times J\in\mathcal{Q}_k} T_{I,J}g_{I,J}$, is bounded from $L^r(\mathbb{R}^n, l^r)$ to $L^r(\mathbb{R}^n)$ for $r=1$ and $r=2$. By the Riesz-Thorin theorem interpolation theorem for such operators, see, e.g., [G1], T_k is also bounded from $L^{q/2}(\mathbb{R}^n, l^{q/2})$ to $L^{q/2}(\mathbb{R}^n)$, since $1 \leq q/2 \leq 2$ (all of course with norms independent of k). Thus

$$\|\sum_{I\times J\in\mathcal{Q}_k} T_{I,J}g_{I,J}\|^{q/2}_{L^{q/2}(\mathbb{R}^n)} \lesssim \sum_{I\times J\in\mathcal{Q}_k} \|g_{I,J}\|^{q/2}_{L^{q/2}(\mathbb{R}^n)}.$$

Observe now that

$$T_{I,J}(\widehat{f_I}\widehat{f_J}) = \widehat{f_I}\widehat{f_J}$$

whence

$$\|\sum_{I\times J\in\mathcal{Q}_k} \widehat{f_I}\widehat{f_J}\|^{q/2}_{L^{q/2}(\mathbb{R}^n)} \lesssim \sum_{I\times J\in\mathcal{Q}_k} \|\widehat{f_I}\widehat{f_J}\|^{q/2}_{L^{q/2}(\mathbb{R}^n)}. \tag{10.6}$$

In order to apply our bilinear assumption we have to scale f_I and f_J back to the unit scale. Let $I \times J \in \mathcal{Q}_k$. After a translation and rotation we may assume that $I \cup J \subset B(0, C2^{-k}) \subset \mathbb{R}^{n-1}$. Then the appropriate scaling is $(v, \varphi(v)) \mapsto (2^k v, 2^{2k}\varphi(v)) := (w, \psi(w))$. Define, with this notation,

$$g_I(w, \psi(w)) = f_I(v, \varphi(v)), g_J(w, \psi(w)) = f_J(v, \varphi(v)).$$

The change of variable formulas give ($\widehat{g_I}$ and $\widehat{g_J}$ are now of course with respect to the graph of ψ)

$$\widehat{f_I}(x,t) = 2^{-k(n-1)}\widehat{g_I}(2^{-k}x, 2^{-2k}t), \widehat{f_J}(x,t) = 2^{-k(n-1)}\widehat{g_J}(2^{-k}x, 2^{-2k}t),$$

$$\int |\widehat{f_I}\widehat{f_J}|^{q/2} = 2^{-k(q(n-1)-(n+1))}\int |\widehat{g_I}\widehat{g_J}|^{q/2},$$

$$\int |g_I|^p = 2^{k(n-1)}\int |f_I|^p, \int |g_I|^p = 2^{k(n-1)}\int |f_I|^p.$$

The vaguely stated assumption of the theorem includes that (10.3) holds for a class of surfaces containing the graph of ψ (and Tao's Theorem 10.2 also includes these surfaces). Therefore

$$\|\widehat{g_I}\widehat{g_J}\|^{q/2}_{L^{q/2}(\mathbb{R}^n)} \lesssim \|g_I\|^{q/2}_{L^p(S^{n-1})}\|g_J\|^{q/2}_{L^p(S^{n-1})}.$$

Combining these statements we find

$$\|\widehat{f_I}\widehat{f_J}\|^{q/2}_{L^{q/2}(\mathbb{R}^n)} \lesssim 2^{-k(\frac{(n-1)}{p'} - \frac{(n+1)}{q})q}\|f_I\|^{q/2}_{L^p(S^{n-1})}\|f_J\|^{q/2}_{L^p(S^{n-1})}.$$

Recalling (10.4), inserting the last estimate into (10.6), and using the fact that for each I there are only boundedly many J's such that $I \times J \in \mathcal{Q}_k$, we obtain

$$\begin{aligned}\|\hat{f}\|^2_{L^q(\mathbb{R}^n)} &\lesssim \sum_k \Big(\sum_{I\times J\in\mathcal{Q}_k} 2^{-k(\frac{(n-1)}{p'} - \frac{(n+1)}{q})q}\|f_I\|^{q/2}_{L^p(S^{n-1})}\|f_J\|^{q/2}_{L^p(S^{n-1})}\Big)^{2/q}\\ &\lesssim \sum_k 2^{-2k(\frac{(n-1)}{p'} - \frac{(n+1)}{q})}\Big(\sum_{I\in\mathcal{D}_k}\|f_I\|^q_{L^p(S^{n-1})}\Big)^{2/q}.\end{aligned}$$

The summation in I is over all dyadic subcubes of Q of side-length 2^{-k}. The factor $\frac{(n-1)}{p'} - \frac{(n+1)}{q}$ is positive by our assumptions. So the theorem follows if we have

$$\sum_{I\in\mathcal{D}_k}\|f_I\|^q_{L^p(S^{n-1})} \leq \|f\|^q_{L^p(S^{n-1})}.$$

This is true if $q/p \geq 1$. Choosing p' sufficiently close to $\frac{n-1}{n+1}q$ we do have $q/p \geq 1$ due to the assumption $q > 2n/(n-1)$; if $p' = \frac{n-1}{n+1}q$, $q/p = \frac{(n-1)q-n-1}{n-1} > 1$. Moreover, getting the result for some p gives it also for larger p (and smaller p').

10.2. **Localization.** We now proceed towards the proof of Tao's Theorem 10.2. The main ideas are due to Wolff who proved first the analogous sharp result for the cone. The first step is to reduce to local estimates.

The following theorem is due to Tao and Vargas, see [TV]. There is also a version for one function which was proved earlier by Bourgain. The relations between p and q are probably not sharp, but all that is really needed is that if the assumption holds for all $\alpha > 0$, then the assertion holds for all $p > q$.

Theorem 10.5. Let $S_j \subset S^{n-1}, j = 1,2$, with $\operatorname{dist}(S_1, S_2) \gtrsim 1$ and $f_j \in \mathcal{S}$ with $\operatorname{spt} f_j \subset S_j$. Suppose that $1 < p < \frac{n+1}{n-1}, \alpha > 0$ and $\frac{1}{p}(1+\frac{4\alpha}{n-1}) < \frac{1}{q} + \frac{2\alpha}{n+1}$. If

$$||\widehat{f_1}\widehat{f_2}||_{L^q(B(x,R))} \lesssim_\alpha R^\alpha ||f_1||_2||f||_2 \text{ for } x \in \mathbb{R}^n, R > 1, f_j \in L^2(S_j), j = 1,2, \tag{10.7}$$

then

$$||\widehat{f_1}\widehat{f_2}||_{L^p(\mathbb{R}^n)} \lesssim ||f_1||_2||f||_2 \text{ for } f_j \in L^2(S_j), j = 1,2. \tag{10.8}$$

From now on we shall assume that S_j is the graph of a smooth function φ_j. We shall denote by $A(r)$ the r-neighbourhood $\{x : \operatorname{dist}(x,a) < r\}$ of a set A.

The Fourier transform of the surface measure σ^{n-1} satisfies

$$\widehat{\sigma^{n-1}}(x) \lesssim (1+|x|)^{-\beta}, x \in \mathbb{R}^n, \text{ where } \beta = (n-1)/2.$$

Our assumptions on p and q in terms of β read

$$1 < p < \frac{\beta+1}{\beta}, \frac{1}{p}(1+\frac{2\alpha}{\beta}) < \frac{1}{q} + \frac{\alpha}{1+\beta}.$$

The proof of Theorem 10.5 will be based on three lemmas. The first of these says that the hypothesis (10.7) yields a similar statement if the functions live in neighbourhoods of the surface.

Lemma 10.6. (a) If C is a positive number and μ a Borel measure on $\mathbb{R}^n$ such that

$$||\widehat{f}||_{L^q(\mu)} \leq C||f||_2 \text{ for } f \in L^2(S_j), j = 1,2, \tag{10.9}$$

then for all $r > 0$,

$$||\widehat{f}||_{L^q(\mu)} \lesssim C\sqrt{r}||f||_2 \text{ for } f \in L^2(S_j(r)), \tag{10.10}$$

where $S_j(r)$ is the r-neighborhood of S_j.
(b) If C is a positive number and μ a Borel measure on $\mathbb{R}^n$ such that

$$||\widehat{f_1}\widehat{f_2}||_{L^q(\mu)} \leq C||f_1||_2||f_2||_2 \text{ for } f_j \in L^2(S_j), j = 1,2, \tag{10.11}$$

then for all $r > 0$,

$$||\widehat{f_1}\widehat{f_2}||_{L^q(\mu)} \lesssim Cr||f_1||_2||f_2||_2 \text{ for } f_1 \in L^2(S_1(r)), f_2 \in L^2(S_2(r)). \tag{10.12}$$

Proof. Let $S_{j,t} = \{(x, \varphi_j(x) + t) : x \in V_j\}$ and $S_j^r = \cup_{|t|<r} S_{j,t}$. It is enough to prove the lemma for S_j^r in place of $S_j(r)$ since $S_j(r/C_1) \subset S_j^r \subset S_j(C_1 r)$ for some constant C_1.

Let $f_j \in L^2(S_j^r)$ and denote $f_{j,t}(z) = f_j(x, \varphi_j(x) + t)$ for $z = (x, \varphi_j(x)) \in S_j$. Then by Fubini's theorem,

$$|\widehat{f_j}(x,t)| = |\int\int e^{-2\pi i(x\cdot y + ts)} f_j(y,s)dsdy| = |\int_{-r}^{r}\int e^{-2\pi i(x\cdot y + t(u+\varphi_j(y)))} f_{j,u}(y, \varphi_j(y))dydu|$$

$$= |\int_{-r}^{r} e^{-2\pi itu}\widehat{f_{j,u}}(x,t)du| \leq \int_{-r}^{r} |\widehat{f_{j,u}}(x,t)|du.$$

Thus using Minkowski's integral inequality, (10.11), Schwartz's inequality and Fubini's theorem,

$$
\begin{aligned}
||\widehat{f}_1\widehat{f}_2||_{L^q(\mu)} &\leq (\int |\int_{-r}^{r} \widehat{f_{1,u}}(z)du \int_{-r}^{r} \widehat{f_{2,v}}(z)dv|^q d\mu z)^{1/q} \\
&= (\int |\int_{-r}^{r}\int_{-r}^{r} \widehat{f_{1,u}}(z)\widehat{f_{2,v}}(z)dudv|^q d\mu z)^{1/q} \\
&\leq \int_{-r}^{r}\int_{-r}^{r} \|\widehat{f_{1,u}}\widehat{f_{2,v}}\|_{L^q(\mu)}dudv \\
&\leq C\int_{-r}^{r}\int_{-r}^{r} ||f_{1,u}||_2||f_{2,v}||_2 dudv \\
&\leq C\sqrt{2r}(\int_{-r}^{r}\int_{S_1} |f_{1,u}|^2du)^{1/2}\sqrt{2r}(\int_{-r}^{r}\int_{S_2} |f_{2,v}|^2dv)^{1/2} \\
&\approx 2Cr||f_1||_2||f_2||_2.
\end{aligned}
$$

□

The second lemma shows that a local hypothesis, namely (10.13), gives a global estimate for functions living in neighbourhoods of the surfaces S_j.

Lemma 10.7. Let C and R be positive numbers such that

$$||\widehat{f}_1\widehat{f}_2||_{L^q(B(x,R))} \lesssim C||f_1||_2||f||_2 \text{ for } x \in \mathbb{R}^n, f_j \in L^2(S_j(2/R)), j = 1, 2, \tag{10.13}$$

then

$$||\widehat{f}_1\widehat{f}_2||_{L^q(\mathbb{R}^n)} \lesssim C||f_1||_2||f_2||_2 \text{ for } f_j \in L^2(S_j(1/R)), j = 1, 2. \tag{10.14}$$

Proof. Let $f_j \in L^2(S_j(1/R)), j = 1, 2$. Let $\psi \in \mathcal{S}$ be such that $0 \leq \psi \leq 1, \psi \approx 1$ on $B(0, 1)$ and $\check{\psi} \subset B(0, 1)$. Cover $\mathbb{R}^n$ with balls $B(x_k, R/2), k = 1, 2, \ldots$, such that $\sum_k \chi_{B(x_k,R)} \approx 1$. Define $\psi_k(x) = \psi((x - x_k)/R)$. Then $\sum_k \psi_k \approx \sum_k \psi_k^{2q} \approx \sum_k \chi_{B(x_k,R)}\psi_k^{2q} \approx 1$. Moreover $\operatorname{spt}\check{\psi}_k \subset B(0, 1/R)$, whence $\operatorname{spt}\check{\psi}_k * f_j \subset S_j(2/R)$. Applying (10.13) and Plancherel's theorem, we obtain

$$
\begin{aligned}
\|\psi_k^2\widehat{f}_1\widehat{f}_2\|_{L^q(B(x_k,R))} &= \|\widehat{\check{\psi}_k * f_1}\widehat{\check{\psi}_k * f_2}\|_{L^q(B(x_k,R))} \\
&\lesssim C\|\check{\psi}_k * f_1\|_2\|\check{\psi}_k * f_2\|_2 = C\|\psi_k\widehat{f}_1\|_2\|\psi_k\widehat{f}_2\|_2.
\end{aligned}
$$

Summing over k we get by Schwartz's inequality,

$$
\begin{aligned}
\|\widehat{f}_1\widehat{f}_2\|_{L^q(\mathbb{R}^n)} &\lesssim \sum_k \|\chi_{B(x_k,R)}\psi_k^2\widehat{f}_1\widehat{f}_2\|_{L^q(\mathbb{R}^n)} \\
&= \|\psi_k^2\widehat{f}_1\widehat{f}_2\|_{L^q(B(x_k,R))} \lesssim C\sum_k \|\psi_k\widehat{f}_1\|_2\|\psi_k\widehat{f}_2\|_2 \\
&\leq C(\sum_k \|\psi_k\widehat{f}_1\|_2^2)^{1/2}(\sum_k \|\psi_k\widehat{f}_2\|_2^2)^{1/2} \\
&\approx C\|\widehat{f}_1\|_2\|\widehat{f}_2\|_2.
\end{aligned}
$$

□

Corollary 10.8. Assuming (10.7) we have for all $R > 1$,

(10.15) $$||\widehat{f_1}\widehat{f_2}||_{L^q(\mathbb{R}^n)} \lesssim R^{\alpha-1}||f_1||_2||f_2||_2 \text{ for } f_j \in L^2(S_j(1/R)), j = 1, 2.$$

Proof. Applying the assumption (10.7) and Lemma 10.6 with $\mu = \mathcal{L}^n\lfloor B(x, r)$ we have

$$||\widehat{f_1}\widehat{f_2}||_{L^q(B(x,R))} \lesssim R^{\alpha-1}||f_1||_2||f||_2 \text{ for } x \in \mathbb{R}^n, f_j \in L^2(S_j(2/R)), j = 1, 2.$$

Hence the corollary follows by Lemma 10.7. □

The third lemma tells us how estimates on functions defined in neighborhoods of the surfaces S_j lead to estimates for functions defined on the surfaces themselves.

Lemma 10.9. For any $F \in L^\infty \cap L^1$ with $\|F\|_\infty \leq 1$ and any $N, R > 1$,

(10.16) $$|\int F\widehat{g_1}\widehat{g_2}|^2 \lesssim_N R^{-\beta}\|F\|^2_{\frac{2\beta+2}{\beta+2}} + \sum_{k=0}^{\infty} R2^{-kN}\|\widehat{F\widehat{g_2}}\|^2_{L^2(S_1(2^k/R))},$$

for all $g_j \in L^2(S_j)$ with $\|g_j\|_{L^2(S_j)} \leq 1$, and

(10.17) $$|\int F\widehat{g_1}\widehat{h_2}|^2 \lesssim_N \lambda R^{-\beta-1}\|F\|^2_{\frac{2\beta+2}{\beta+2}} + \sum_{k=0}^{\infty} R2^{-kN}\|\widehat{F\widehat{h_2}}\|^2_{L^2(S_1(2^k/R))},$$

for all $g_1 \in L^2(S_1), h_2 \in L^2(S_2(\lambda/R)), \lambda > 0$ with $\|g_1\|_{L^2(S_1)} \leq 1, \|h_2\|_2 \leq 1$.

Proof. By the product formula, Schwartz's inequality and Plancherel's theorem, ,

$$\begin{aligned}|\int F\widehat{g_1}\widehat{g_2}|^2 &= |\int \widehat{F\widehat{g_2}}g_1 d\sigma^{n-1}|^2 \\ &\leq \|\widehat{F\widehat{g_2}}\|^2_{L^2(S_1)}\|g_1\|^2_{L^2(S_1)} \leq \int ((F\widehat{g_2}) * \widehat{\sigma^{n-1}})\overline{F\widehat{g_2}}.\end{aligned}$$

By Hölder's inequality and the Stein-Tomas restriction theorem, Theorem 4.4,

(10.18) $$\|F\widehat{g_2}\|_1 \leq \|F\|_{\frac{2\beta+2}{\beta+2}}\|\widehat{g_2}\|_{\frac{2\beta+2}{\beta}} \lesssim \|F\|_{\frac{2\beta+2}{\beta+2}}.$$

Choose $\varphi \in \mathcal{S}$ such that $\varphi = 1$ on $B(1)$, φ vanishes outside $B(2)$ and write σ^{n-1} as

$$\sigma^{n-1} = \tau_1 + \tau_2 \text{ with } \widehat{\tau_1}(x) = \varphi(x/R)\widehat{\sigma_1}(x).$$

Then

$$\|\widehat{\tau_2}\|_\infty \lesssim R^{-\beta},$$

and so by (10.18)

(10.19) $$\int ((F\widehat{g_2}) * \widehat{\tau_2})\overline{F\widehat{g_2}} \leq \|\widehat{\tau_2}\|_\infty\|F\widehat{g_2}\|^2_1 \leq R^{-\beta}\|F\|^2_{\frac{2\beta+2}{\beta+2}}.$$

Next we estimate $|\int((F\widehat{g_2})*\widehat{\tau_1})\overline{F\widehat{g_2}}|$. By Plancherel's theorem (note that $\tau_1 \in L^2$)

$$|\int ((F\widehat{g_2}) * \widehat{\tau_1})\overline{F\widehat{g_2}}| \lesssim \int |\widehat{F\widehat{g_2}}|^2|\tau_1|.$$

Using the rapid decay of $\hat{\varphi}$ one checks easily that

$$|\tau_1(x)| = |R^n \int \hat{\varphi}(R(y-x))d\sigma_1 y| \lesssim_N R(1+d(x,S_1))^{-N}.$$

Hence

$$|\int ((F\widehat{g_2}) * \widehat{\tau_1})\overline{F\widehat{g_2}}| \lesssim_N \sum_{k=0}^{\infty} R2^{-kN} \|\widehat{F\widehat{g_2}}\|^2_{L^2(S_1(2^k/R))}.$$

This proves (10.16). To prove (10.17) we argue in the same way but use Stein-Tomas theorem in combination with Lemma 10.6(a) to have

$$\|\widehat{h_2}\|_{\frac{2\beta+2}{\beta}} \lesssim \lambda^{1/2}R^{-1/2}.$$

□

In order to complete the proof of Theorem 10.5 we shall prove that for any measurable set $A \subset \mathbb{R}^n$ with $1 \leq \mathcal{L}^n(A) < \infty$,

$$||\chi_A \widehat{g_1}\widehat{g_2}||_{L^1(\mathbb{R}^n)} \lesssim \mathcal{L}^n(A)^{1/p'}||g_1||_2||g||_2 \text{ for } g_j \in L^2(S_j), j = 1,2. \tag{10.20}$$

Let us first see how this implies the theorem. Fix $f_j \in L^2(S_j), j = 1,2$, with $\|f_j\|_{L^2(S_j)} = 1$. Apply (10.20) with

$$A = \{x : |\widehat{f_1}\widehat{f_2}(x)| > \lambda\}, \lambda > 0.$$

Note that $\mathcal{L}^n(A) < \infty$ because $\widehat{f_j} \in L^{p_0}$ for some $p_0 < \infty$ by the Stein-Tomas restriction theorem. Then by (10.20), if $\mathcal{L}^n(A) \geq 1$,

$$\lambda\mathcal{L}^n(A) \leq ||\chi_A \widehat{f_1}\widehat{f_2}||_{L^1(\mathcal{L}^n)} \lesssim \mathcal{L}^n(A)^{1/p'}$$

which gives

$$\mathcal{L}^n(\{x : |\widehat{f_1}\widehat{f_2}(x)| > \lambda\}) \lesssim \max\{\lambda^{-p}, 1\}.$$

Combining this weak type inequality with the trivial inequality

$$\|\widehat{f_1}\widehat{f_2}\|_\infty \lesssim 1,$$

(10.8) follows by interpolation.

Applying Lemma 10.9 with $N = 3$ and choosing (notice that the exponent below is positive as $1 < p < \frac{\beta+1}{\beta}$)

$$R = \mathcal{L}^n(A)^{\frac{1}{\beta}(\frac{\beta+2}{\beta+1} - \frac{2}{p'})}. \tag{10.21}$$

we obtain

$$|\int \chi_A \widehat{g_1}\widehat{g_2}|^2 \lesssim R^{-\beta}\mathcal{L}^n(A)^{\frac{\beta+2}{\beta+1}} + \sum_{k=0}^{\infty} R2^{-3k}\|\widehat{\chi_A\widehat{g_2}}\|^2_{L^2(S_1(2^k/R))}$$

$$= \mathcal{L}^n(A)^{2/p'} + \sum_{k=0}^{\infty} R2^{-3k}\|\widehat{\chi_A\widehat{g_2}}\|^2_{L^2(S_1(2^k/R))}.$$

Here

$$\|\widehat{\chi_A\widehat{g_2}}\|_{L^2(S_1(2^k/R))} = \sup_{\|h_{1,k}\|_{L^2(S_1(2^k/R))}\leq 1} \int \widehat{\chi_A\widehat{g_2}}h_{1,k}$$
$$= \sup_{\||h_{1,k}\|_{L^2(S_1(2^k/R))}\leq 1} \int \chi_A\widehat{g_2}\widehat{h_{1,k}}.$$

We can repeat the above argument with g_2 playing the role of g_1 and $h_{1,k}$ playing the role of g_2. Now $h_{1,k}$ is in $L^2(S_1(2^k/R))$ with norm at most 1 and we have by (10.17)

$$|\int \chi_A\widehat{g_2}\widehat{h_{1,k}}|^2 \lesssim 2^k R^{-1}\mathcal{L}^n(A)^{2/p'} + \sum_{l=0}^{\infty} R2^{-3l}\|\widehat{\chi_A\widehat{h_{1,k}}}\|^2_{L^2(S_1(2^l/R))};$$

Again

$$\|\widehat{\chi_A\widehat{h_{1,k}}}\|_{L^2(S_1(2^lR))} = \sup_{\|h_{2,l}\|_{L^2(S_2(2^l/R))}\leq 1} \int \widehat{\chi_A\widehat{h_{1,k}}}h_{2,l}$$
$$\leq \sup_{\|h_{2,l}\|_{L^2(S_2(2^l/R))}\leq 1} \|\chi_A\widehat{h_{1,k}}\widehat{h_{2,l}}\|_{L^1(\mathbb{R}^n)}.$$

By Hölder's inequality

$$\|\chi_A\widehat{h_{1,k}}\widehat{h_{2,l}}\|_{L^1(\mathbb{R}^n)} \leq \mathcal{L}^n(A)^{1/q'}\|\widehat{h_{1,k}}\widehat{h_{2,l}}\|_{L^q(\mathbb{R}^n)}.$$

By Corollary 10.8 we have for $k \leq l$,

$$\|\widehat{h_{1,k}}\widehat{h_{2,l}}\|_{L^q(\mathbb{R}^n)} \lesssim R^{\alpha-1}2^l.$$

Combining these inequalities,

$$\|\chi_A\widehat{g_1}\widehat{g_2}\|^2_{L^1(\mathbb{R}^n)}$$
$$\lesssim \mathcal{L}^n(A)^{2/p'} + \sum_{k=0}^{\infty} R2^{-3k}\sum_{l=0}^{\infty} R2^{-3l}\mathcal{L}^n(A)^{2/q'}R^{2\alpha-2}2^{2l}$$
$$\approx \mathcal{L}^n(A)^{2/p'} + \mathcal{L}^n(A)^{2/q'}R^{2\alpha}.$$

Recalling how we chose R in (10.21) we see that

$$\mathcal{L}^n(A)^{2/q'}R^{2\alpha} = \mathcal{L}^n(A)^{2/q'+\frac{2\alpha}{\beta}(\frac{\beta+2}{\beta+1}-\frac{2}{p'})}.$$

Since

$$2/q' + \frac{2\alpha}{\beta}(\frac{\beta+2}{\beta+1} - \frac{2}{p'}) < 2/p'$$

the desired inequality (10.20) follows and the proof of the theorem is complete.

10.3. **Induction on scales.** The second crucial idea is an induction on scales argument due to Wolff. That is, (10.7) is reduced to

Proposition 10.10. There is a constant $c > 0$ such the following holds. Suppose that (10.7) holds for some $\alpha > 0$:

(10.22) $||\widehat{f_1}\widehat{f_2}||_{L^q(B(x,R))} \lesssim_\alpha R^\alpha ||f_1||_2||f_2||_2$ for $x \in \mathbb{R}^n, R > 1, f_j \in L^2(S_j), j = 1, 2.$

Then for all $0 < \delta, \varepsilon < 1$ there exists a constant C such that
(10.23)
$$||\widehat{f_1}\widehat{f_2}||_{L^q(B(x,R))} \leq CR^{\max\{\alpha(1-\delta),c\delta\}+\varepsilon}||f_1||_2||f_2||_2, x \in \mathbb{R}^n, R > 1, \operatorname{spt} f_j \subset S_j.$$

Then Proposition 10.10 implies (10.7) for all $\alpha > 0$. To see this note that $||\widehat{f_j}||_\infty \lesssim ||f_j||_2$ by Schwartz's inequality, whence (10.7) holds for $\alpha = \alpha_0 = s/q$. Fix $\varepsilon > 0$ and define

$$\alpha_{j+1} = c\alpha_j/(\alpha_j + c) + \varepsilon, j = 0, 1, 2, \ldots,$$

Suppose (10.7) holds for $\alpha = \alpha_j$ for some j. Apply Proposition 10.10 with $\delta = \delta_j = \alpha_j/(\alpha_j + c)$. Then

$$\max\{\alpha_j(1-\delta), c\delta\} = c\alpha_j/(\alpha_j + c),$$

and it follows that (10.7) holds for $\alpha = \alpha_{j+1}$. It is easy to check that

$$\alpha_j \to (\varepsilon + \sqrt{\varepsilon^2 + 4c\varepsilon})/2.$$

Since we can choose ε arbitrarily small, (10.7) holds for all $\alpha > 0$.

10.4. **Wavepacket decomposition.** The proof of Proposition 10.10, which is the core of the whole argument, uses the third basic tool: the wavepacket decomposition. Fix $R > 1$ and let again $S_j = \{(v, \varphi_j(v)) : v \in V_j\} \subset S^{n-1}, V_j \subset \mathbb{R}^{n-1}, j = 1, 2,$ be such that $\operatorname{dist}(S_1, S_2) \gtrsim 1$ and let $f_j \in L^2(S_j)$. The wavepacket decomposition allows us to write $\widehat{f_j}$ as a sum of functions p_{y,v_j} which together with their Fourier transforms are well localized:

(10.24) $$\widehat{f_j}(x,t) = \Sigma_{w_j} p_{w_j}(x,t), j = 1, 2.$$

The indices w_j (where w_1 is always related to S_1 and w_2 to S_2) are of the form (y_j, v_j) where v_j's run through a $1/\sqrt{R}$-separated set in V_j and y_j's run through a $\sqrt{R}$-separated set in $\mathbb{R}^{n-1}$. The functions p_{w_j} are essentially supported in the tubes (that is, decay very fast off them)

$$T_{w_j} = \{(x,t) : |t| \leq R, |x - (y_j + t\nabla\varphi_j(v_j))| \leq R^{1/2}\}$$

and their Fourier transforms have supports in $S_j \cap B((v_j, \varphi_j(v_j)), 2/\sqrt{R})$. The transversality assumptions on S_1 and S_2 guarantee that any two tubes T_{w_1} and T_{w_2} are transversal.

The proof of the wavepacket decomposition involves several technicalities, but in principle it is not very difficult. Here are the main ideas: First find (using the so-called Poisson summation formula) the C^∞-functions η and ψ on $\mathbb{R}^{n-1}$ such that

$$\operatorname{spt}\widehat{\eta} \subset B(0,1), \operatorname{spt}\psi \subset B(0,1), \sum_{k\in\mathbb{Z}^{n-1}} \eta(x-k) = \sum_{k\in\mathbb{Z}^{n-1}} \psi(x-k) = 1 \text{ for } x \in \mathbb{R}^{n-1}.$$

Define for $y_j \in R^{1/2}\mathbb{Z}^{n-1}$ and $v_j \in R^{-1/2}\mathbb{Z}^{n-1} \cap V_j$,

$$\eta_{y_j}(x) = \eta(\frac{x+y_j}{\sqrt{R}}), \psi_{v_j}(v) = \psi(\sqrt{R}(v-v_j)), x, v \in \mathbb{R}^{n-1}.$$

Then

$$\widehat{\eta_{y_j}}(v) = R^{(n-1)/2}e^{2\pi i v\cdot y_j}\widehat{\eta}(\sqrt{R}v), \operatorname{spt}\widehat{\eta_{y_j}} \subset B(0,1/\sqrt{R}), \operatorname{spt}\psi_{v_j} \subset B(v_j,1/\sqrt{R}).$$

Defining g_j on V_j by $g_j(v) = f_j(v,\varphi(v))$, we have

$$\sum_{y_j} \eta_{y_j} = 1 \text{ and } g_j = \sum_{v_j} \psi_{v_j} g_j.$$

Thus

$$g_j = \sum_{v_j,y_j} \mathcal{F}^{-1}(\widehat{\psi_{v_j}g_j}\eta_{y_j}).$$

Now the functions p_{y,v_j}

$$p_{y_j,v_j}(x,t) = \int_{V_j} e^{2\pi i(x\cdot v - t\varphi_j(v))}\mathcal{F}^{-1}(\widehat{\psi_{v_j}g_j}\eta_{y_j})(v)dv, (x,t) \in \mathbb{R}^{n-1}\times\mathbb{R},$$

have the required properties. The decomposition $\widehat{f_j}(x,t) = \Sigma_{w_j} p_{w_j}(x,t)$ and the fact $\widehat{\operatorname{spt} p_{w_j}} \subset S_j \cap B((v_j,\varphi_j(v_j)), 2/\sqrt{R})$ are easily checked. The fast decay of p_{w_j} outside T_{w_j} follows by stationary phase estimates, more precisely, by Theorem 3.4.

10.5. **The final geometric and combinatorial estimates.** In order to prove Proposition 10.2, and thus complete the proof of Theorem 10.10, we need, by (10.24), the estimate

$$\|\Sigma_{w_j} p_{w_1}p_{w_2}\|_{L^q(Q(R))} \lesssim R^\varepsilon(R^{(1-\delta)\alpha} + R^{c\delta}).$$

Here $Q(R)$ is the cube of sidelength R centered at the origin. Some pigeonholing arguments and normalizations of the functions p_{w_j} reduce this to

$$\|\Sigma_{w_j\in W_j} p_{w_1}p_{w_2}\|_{L^q(Q(R))} \lesssim R^\varepsilon(R^{(1-\delta)\alpha} + R^{c\delta})\sqrt{\sharp W_1 \sharp W_2}$$

for arbitrary subsets W_j of the index sets under the conditions

$$\|p_{w_j}\|_\infty \lesssim R^{(1-n)/4}.$$

Next the cube $Q(R)$ is decomposed into cubes $Q \in \mathcal{Q}$ of side-length $R^{1-\delta}$, a relation $w_j \sim Q$ is defined and the above sum is split to the local part; $w_1 \sim Q$ and $w_2 \sim Q$, and the far-away part; $w_1 \not\sim Q$ or $w_2 \not\sim Q$. Local here means that for a given w_j the cubes Q with $w_j \sim Q$ are contained is some cube with side-length $\approx R^{1-\delta}$ which allows us to use the induction hypothesis (10.22) to get the upper bound $R^\varepsilon R^{(1-\delta)\alpha}\sqrt{\sharp W_1 \sharp W_2}$ for this part of the sum.

The far-away part will be estimated by $R^{\varepsilon}R^{c\delta}\sqrt{\sharp W_1 \sharp W_2}$. First there is the L^1-estimate

$$\|\sum_{w_1\in W_1, w_2\in W_2, w_1\not\sim Q \text{ or } w_2\not\sim Q} p_{w_1}p_{w_2}\|_{L^1(Q)} \lesssim R(\#W_1)^{1/2}(\#W_2)^{1/2},$$

which follows by some L^2-estimates for the functions p_{w_j}. Hence by interpolation the required estimate is reduced to showing that for every $Q \in \mathcal{Q}$,

$$\|\sum_{w_1\in W_1, w_2\in W_2, w_1\not\sim Q \text{ or } w_2\not\sim Q} p_{w_1}p_{w_2}\|_{L^2(Q)} \lesssim R^{c\delta-(n-2)/4}(\#W_1)^{1/2}(\#W_2)^{1/2}.$$

Next $Q(R)$ is split into cubes $P \in \mathcal{P}$ of side-length $\sqrt{R}$. We are lead show that for any $Q \in \mathcal{Q}$,

$$\sum_{P\in\mathcal{P}, P\subset 2Q} \|\sum_{R^{\delta}P\cap T_{w_j}\neq\varnothing, w_1\not\sim Q \text{ or } w_2\not\sim Q} p_{w_1}p_{w_2}\|^2_{L^2(P)} \lesssim R^{c\delta-(n-2)/2}(\#W_1)(\#W_2).$$

The reduction to $R^{\delta}P \cap T_{w_j} \neq \varnothing$ follows from the fast decay of p_{w_j} outside T_{w_j}. The far-away porperty involves that we can sum over such w_j that when $P_0 \subset 2Q$ there are many $P \in \mathcal{P}$ for which $R^{\delta}P \cap T_{w_j} \neq \varnothing$. Writing

$$\|\sum_{w_1\in U_1, w_2\in U_2} p_{w_1}p_{w_2}\|_2^2 = \sum_{w_1, w_1'\in U_1, w_2'\in U_2} \int p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}},$$

and

$$\int p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}} = \int \widehat{p_{w_1}p_{w_2}}\,\overline{\widehat{p_{w_1'}p_{w_2'}}} = \int (\widehat{p_{w_1}} * \widehat{p_{w_2}})\overline{\widehat{p_{w_1'}} * \widehat{p_{w_2'}}},$$

the support properties $\widehat{p_{w_j}}$'s are used to estimate

$$|\int p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}}| \lesssim R^{-(n-2)/2}.$$

Furthermore, the support properties yield that if we fix w_1 and w_2' and if w_1' is such that $\int p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}} \neq 0$ for some w_2, then v_1' lies in an $R^{-1/2}$-neighborhood of a smooth hypersurface depending on w_1 and w_2'. The geometry of this surface is well understood because of the initial transversality and curvature assumptions for the surfaces S_j. For the functions φ_j they require that the directions of the gradients are separated and also a non-degeneracy condition of the Hessians. These and the transversality of the tubes T_{w_1} and T_{w_2} lead to good estimates on the number of indices for which $\int p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}} \neq 0$ completing the proof.

10.6. **Multilinear restriction and improvements by Bourgain and Guth.** In [BCT] Bennet, Carbery and Tao proved multilinear restriction and Kakeya estimates. For example, in three dimension they proved

$$\|\hat{f}_1\hat{f}_2\hat{f}_3\|_{L^q(\mathbb{R}^3)} \lesssim \|f_1\|_{L^\infty(S_1)}\|f_2\|_{L^\infty(S_2)}\|f_3\|_{L^\infty(S_3)}$$

provided $q > 3$ and the normals n_1, n_2, n_3 of the surfaces S_1, S_2, S_3 are never close to a two-dimensional plane. Bourgain and Guth used this, together with Kakeya arguments, to improve the restriction estimates to

$$\|\hat{f}\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^\infty(S^2)} \text{ for } f \in L^\infty(S^2), p > 33/10.$$

Recall that Tao's bilinear estimate and Theorem 10.4 gave this for $p > 10/3$ so there is an improvement by $1/30$. Both papers deal with many other aspects and in general dimensions.

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