4^{th} Sheet of Exercise

23^{rd} February 2012

Notation. Along this sheet, we will follow the following notation. If X is an open subset of $\mathbb{R}^m$ with m a positive integer, then $C^\infty(X)$ is the space of smooth functions in X. Finally, $\mathcal{S}(\mathbb{R}^m)$ denotes the space of rapidly decreasing smooth functions.

Exercises. Along these exercises we consider $a \in \mathcal{S}(\mathbb{R}^{2n})$ and $u \in \mathcal{S}(\mathbb{R}^n)$. Set

$$Op(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, \mathrm{d}y\, \mathrm{d}\theta.$$

Set $S^0_{0,0} = \{a \in C^\infty(\mathbb{R}^{2n}) : \partial_x^\alpha \partial_\theta^\beta a \in L^\infty(\mathbb{R}^{2n})\, \forall (\alpha, \beta) \in \mathbb{N}^{2n}\}$.

1. Let $a \in S^0_{0,0}$ and $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\chi(0) = 1$. Show that

   $$\lim_{\varepsilon \to 0} Op(\chi(\varepsilon(\cdot, \cdot))a)u$$

   exists for all $u \in \mathcal{S}(\mathbb{R}^n)$ and defines a linear continuous operator $Op(a) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$. Show that $Op(a)$ can be extended to a continuous operator from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

2. Calculate $Op(a)$ in the following cases

   $$a(x, \theta) = e^{il_x \cdot x} \qquad a(x, \theta) = e^{il_\theta \cdot \theta} \qquad a(x, \theta) = e^{i(l_x \cdot x + l_\theta \cdot \theta)}$$

   where $l_\theta \in \mathbb{R}^n$ and $l_x \in \mathbb{R}^n$.

3. For $n = 1$, $a \in C^\infty(\mathbb{R}^2)$ and 2π-periodic in (x, θ), let

   $$\hat{a}_{j,k} = (2\pi)^{-2} \int \int e^{-i(jx+k\theta)} a(x, \theta)\, \mathrm{d}x \mathrm{d}\theta, \qquad j, k \in \mathbb{Z},$$

   be the Fourier coefficients of a. Show that $Op(a) : L^2 \longrightarrow L^2$ is bounded and that

   $$\|Op(a)\| \leq \sum_{j,k} |\hat{a}_{j,k}|.$$

4. For $a \in \mathcal{S}(\mathbb{R}^2)$ (or even for $a \in S^0_{0,0}$ with $\hat{a} \in L^1$) show that $Op(a)$ is bounded from L^2 to L^2 and

$$\|Op(a)\| \leq \frac{1}{(2\pi)^2} \|\hat{a}\|_{L^1} .$$

5. Show that, for $a \in \mathcal{S}(\mathbb{R}^{2n})$ and $\mathcal{F}$ denoting the Fourier transform,

$$\mathcal{F}^{-1} Op(a) \mathcal{F} = Op(b)$$

where $b(x,\xi) = a \circ \kappa_{\mathcal{F}}(x,\xi)$ with $\kappa_{\mathcal{F}}(x,\xi) = (\xi, -x)$.

6. Let a and b belong to $\mathcal{S}(\mathbb{R}^{2n})$. Show that $Op(a)\, Op(b) = Op(c)$ with

$$c(x,\xi) = \left(e^{i\sigma(D_x, D_\xi; D_y, D_\eta)/2} a(x,\xi) b(y,\eta) \right) \Big|_{\substack{y=x \\ \eta=\xi}} .$$

Here $\sigma(x,\xi;y,\eta) = \xi \cdot y - x \cdot \eta$.

Comments.

(i) These exercises continue with the Weyl quantization started in the previous sheet of exercises.