8^{th} Sheet of Exercise

9^{th} May 2012

Notation. If f, g are two C^1 functions defined on the same open subset in T^*X, we define their Poisson bracket as the continuous function

$$\{f, g\} = H_f(g) = \langle H_f, dg \rangle = \sigma(H_f, H_g).$$

In local coordinates,

$$\{f, g\} = \sum \partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g.$$

Exercises.

1. Let σ be the canonical 2-form on T^*X, and ν a vector field defined on some open subset of T^*X, diffeomorphic to a ball. Show that ν is a Hamilton field if and only if $\mathcal{L}_\nu \sigma = 0$.

2. Show that this three conditions are equivalent:

   (i) $\{u, v\} = 0$ on for all smooth functions u, v on T^*X which vanish on Σ.

   (ii) $T_p\Sigma^\perp \subset T_p\Sigma$ for all $p \in \Sigma$. Here $\perp$ indicates the orthogonal with respect to the canonical 2-form σ.

   (iii) $u = 0$ on Σ implies H_u is tangent to Σ. Here H_u is the Hamiltonian vector field of u.

   A smooth submanifold Σ of T^*X is called involutive if it satisfies (i).

3. Show that if Σ is involutive then $\dim \Sigma \geq \dim X$.

4. Show that Σ is Lagrangian if and only if Σ is involutive and $\dim \Sigma = \dim X$.

5. Let $P \in L^m_{\mathrm{cl}}(\mathbb{R}^n)$ with symbol $p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \cdots + p_{m-j}(x,\xi) + \ldots$, where p_{m-j} is positively homogeneous of degree $m-j$ in ξ. Let us denote $\sigma(P) = p_m$, $\operatorname{sub} P = p_{m-1} - \frac{1}{2i}\sum_{j=1}^n \frac{\partial^2 p_m}{\partial \xi_j \partial x_j}$.

   Show that if $P \in L^m_{\mathrm{cl}}, Q \in L^{m'}_{\mathrm{cl}}$:

$$\operatorname{sub}(P \circ Q) = \sigma(P) \operatorname{sub} Q + \sigma(Q) \operatorname{sub} P + \frac{1}{2i}\{\sigma(P), \sigma(Q)\}.$$

   Here $\{\centerdot, \centerdot\}$ is the Poisson bracket.

6. Let $k \in \mathbb{N}$. Find an expression for $\operatorname{sub} P^k$ where $P^k = P \circ \cdots \circ P$ (k times). What is the degree of homogeneity of $\operatorname{sub} P^k$?