6^{th} Sheet of Exercise

4^{th} April 2012

Notation. Along this sheet, we will follow the following notation. $\mathcal{L}(E,F)$ denotes the space of linear bounded operators from E to B. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing smooth functions. $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ denotes the space of smooth functions in $\mathbb{R}^n \times \mathbb{R}^n$. $L^2(\mathbb{R}^n)$ is the space of measurable function such that their modulus are square integrable functions.

Exercises.

1. Let $A_1, \dots, A_N$ belong to $\mathcal{L}(E,F)$ where E and F are Hilbert spaces. Assume that

$$\sup_{j\in\{1,\dots,N\}} \sum_{k=1}^{n} \left\|A_j^* A_k\right\|^{1/2} \leq M \qquad \sup_{j\in\{1,\dots,N\}} \sum_{k=1}^{n} \|A_j A_k^*\|^{1/2} \leq M.$$

Let $A = \sum_{j=1}^N A_j$.

- Show that $\|A\|^{2m} = \|(A^*A)^m\|$ and that

$$(A^*A)^m = \sum_{j_1,\dots,j_{2m}} A_{j_1}^* A_{j_2} \dots A_{j_{2m-1}}^* A_{j_{2m}}.$$

- Show that

$$\left\|A_{j_1}^* A_{j_2} \dots A_{j_{2m-1}}^* A_{j_{2m}}\right\| \leq$$
$$\leq \left\|A_{j_1}^*\right\|^{1/2} \left\|A_{j_1}^* A_{j_2}\right\|^{1/2} \dots \left\|A_{j_{2m-1}}^* A_{j_{2m}}\right\|^{1/2} \|A_{j_{2m}}\|^{1/2}.$$

- Show that $\|A\|^{2m} \leq NM^{2m}$ for every $m \in \mathbb{N}$ and deduce that $\|A\| \leq M$.

2. Consider now an infinite sequence of operators $A_j \in \mathcal{L}(E, F)$, for $j \in \{1, 2, \dots\}$ such that

$$\sup_{j\in\{1,\dots,N\}} \sum_{k=1}^{\infty} \left\|A_j^* A_k\right\|^{1/2} \le M \qquad \sup_{j\in\{1,\dots,N\}} \sum_{k=1}^{\infty} \|A_j A_k^*\|^{1/2} \le M.$$

Show that $A = \sum_{j=1}^{\infty} A_j$ converges strongly and that the sum A satisfies $\|A\| \le M$. (Consider the series $\sum A_j u$ first when $u \in \Sigma = \sum_k \operatorname{Im} A_k^*$, then when $u \in \overline{\Sigma}$, finally when $u \in \overline{\Sigma}^{\perp}$.)

3. Let $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C_{\alpha,\beta} \forall \alpha, \beta. \tag{1}$$

Consider the pseudodifferentail operator $A \in L^0_{0,0}(\mathbb{R}^n)$ of the form

$$Au(x) = \frac{1}{(2\pi)^n} \int\int e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) \mathrm{d}y \, \mathrm{d}\xi$$

(defined for $u \in \mathcal{S}(\mathbb{R}^n)$ as an iterated integral). We call a the Weyl symbol of A.

Let U_ν be the open ball of radius $R > 0$ and center $\nu \in \mathbb{Z}^{2n}$. Show if R is large enough, there exists a function $\phi_0 \in C_0^\infty(U_0)$ such that the function $\phi_\nu = \phi_0((x, \xi) - \nu) \in C_0^\infty(U_\nu)$ with $\nu \in \mathbb{Z}^{2n}$ form a partition of unity: $1 = \sum_\nu \phi_\nu(x, \xi)$.

4. Show that $a_\nu = a\phi_\nu$ satisfy estimates like (1) with constants $\tilde{C}_{\alpha,\beta}$ and uniform in ν.

5. Let A_ν be a pseudodifferentail operator with Weyl symbol a_ν. Show that $A_\nu A_\mu^*$ has kernel

$$\frac{1}{(2\pi)^n} \int\int\int e^{i(x\cdot\xi - y\cdot\eta)} e^{-iz\cdot(\xi-\eta)} a_\nu\left(\frac{x+z}{2}, \xi\right) \overline{a_\mu}\left(\frac{z+y}{2}, \eta\right) \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}z.$$

Use integrations by parts in ξ, η and z and use Comment 2 to show that, for every N,

$$\left\|A_\nu A_\mu^*\right\|_{\mathcal{L}(L^2,L^2)} \le C_N(1 + |\nu - \mu|)^{-N}.$$

6. Use Exercise 2 and show that A is continuous from L^2 to L^2.

Comments.

1. The result of Exercise 2 is called the Cotlar-Stein lemma.
2. If $K \in C(R^n \times R^n)$ and

$$\sup_y \int |K(x,y)| \, \mathrm{d}x \le C \qquad \sup_x \int |K(x,y)| \, \mathrm{d}y \le C,$$

   then the integral operator A induced by K is bounded in $L^2(R^n)$ and $\|A\| \le C$.
3. The result of Exercise 6 is called the Calderón and Vaillancourt theorem.