5^{th} Sheet of Exercise

28^{th} March 2012

Notation. Along this sheet, we will follow the following notation. $\mathcal{S}(\mathbb{R}^n)$, with n a positive integer, denotes the space of rapidly decreasing smooth functions and $\mathcal{S}'(\mathbb{R}^n)$ stands for the space of tempered distributions. $L^2(\mathbb{R}^n)$ is the space of measurable function such that their modulus are square integrable functions, its associated norm is

$$\|u\|_{L^2}^2 = \int |u|^2 \, \mathrm{d}x.$$

$H^s(\mathbb{R}^n)$ denotes the space of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\langle \cdot \rangle^s \hat{u} \in L^2(\mathbb{R}^n)$.Its norm is defined by

$$\|u\|_{H^s}^2 = \frac{1}{(2\pi)^n} \|\langle \cdot \rangle \hat{u}\|_{L^2}^2 .$$

Here $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. If K is an compact subset of $\mathbb{R}^n$, $\mathcal{E}'(K)$ stands for the space compactly supported distributions such that their support are contained in K.

Exercises.

1. If $s \in \mathbb{N}$, show that $H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : D^\alpha \in L^2(\mathbb{R}^n) \, |\alpha| \leq s\}$ and that for some $C = C(n, s)$:

$$\frac{1}{C} \|u\|_{H^s}^2 \leq \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^2}^2 \leq C \|u\|_{H^s}^2 , \quad u \in H^s(\mathbb{R}^n).$$

2. Let $x = (x', x'')$ with $x' \in \mathbb{R}^{n-d}$ and $x'' \in \mathbb{R}^d$. If $s > d/2$ show that the operator

$$u \in \mathcal{S}(\mathbb{R}^n) \longmapsto u|_{x''=0} \in \mathcal{S}(\mathbb{R}^{n-d}_{x'})$$

can be extended to a bounded operator $H^s(\mathbb{R}^n) \longrightarrow H^{s-d/2}(\mathbb{R}^{n-d})$.

3. Show that the inclusion $H^s(\mathbb{R}^n) \cap \mathcal{E}'(K) \hookrightarrow H^{s'}(\mathbb{R}^n)$ is compact if K is compact and $s > s'$.

   (Hint: Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ in a neighbourhood of K and $u \in \mathcal{E}'(K)$. Then $\chi u = u$ and $\hat{u} = (2\pi)^{-n}\hat{\chi} * \hat{u}$. Show that if $(u_j) \in H^s(\mathbb{R}^n) \cap \mathcal{E}'(K)$ is a bounded sequence, then there exists a subsequence u_{j_ν} converging in $H^{s'}(\mathbb{R}^n)$.)

4. Let a be a smooth positive function in $\mathbb{R}$ and $p(x,\xi) = 1 + a(x_1)\xi_1^2 + i\xi_2$, $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$.

   - Show that for all $\alpha, \beta \in \mathbb{N}^2$ and every compact $K \subset \mathbb{R}^2$, there exists $C = C(K, \alpha, \beta)$ such that
   $$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C|p(x,\xi)|^{1-\beta_1/2-\beta_2}, \quad (x,\xi) \in K \times \mathbb{R}^2.$$
   - Let $q(x,\xi) = 1/p(x,\xi)$. Show that for all $\alpha, \beta \in \mathbb{N}^2$ and every compact $K \subset \mathbb{R}^2$, there exists $C = C(K, \alpha, \beta)$ such that
   $$|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \leq C|p(x,\xi)|^{-1-\beta_1/2-\beta_2}, \quad (x,\xi) \in K \times \mathbb{R}^2.$$

   Deduce that $q \in S^{-1}_{1/2,0}(\mathbb{R}^2 \times \mathbb{R}^2)$.

5. Consider $P(x,D) = 1 - a_1(x)\partial_{x_1}^2 + \partial_{x_2}$ on an open subset Ω of $\mathbb{R}^2$. Let $q(x,D) \in L^{-1}_{1/2,0}(\Omega)$ be properly supported operator with symbol $q(x,\xi)$. Show that $q(x,D) \circ P(x,D) = I - R, \quad R \in L^{-1/2}_{1/2,0}(\Omega)$.

6. Following the same notation:

   - Show that there exists $Q \in L^{-1}_{1/2,0}(\Omega)$ such that $Q(x,D) \circ P(x,D) = I - K$ with $K \in L^{-\infty}(\Omega)$.
   - Let u be a distribution in Ω. What can be said about u if Pu is smooth in Ω? What can be said if $Pu \in H^s_{loc}(\Omega)$?