1^{st} Sheet of Exercise

8^{th} February 2012

Notation. Along this sheet, we will follow the following notation. If X is an open subset of $\mathbb{R}^m$, then $C^\infty(X)$ denotes the space of smooth functions in X. Additionally, $C_0^\infty(X)$ is the subspace of $C^\infty(X)$ such that its elements have compact support in X. We also use the notations $\mathcal{D}'(X)$ for the space of distributions in X and $\mathcal{E}'(X)$ for the subspace of $\mathcal{D}'(X)$ such that its elements have compact support in X.

Exercises.

1. Let $\xi = (\xi', \xi_n)$ belong to $\mathbb{R}^{n-1} \times \mathbb{R}$. Write $|\xi'|^2 = \sum_{j=1}^{n-1} \xi_j^2$ and $|\xi|^2 = |\xi'|^2 + \xi_n^2$. To which symbol spaces do the following symbols belong?

   (a) $(|\xi'|^2 + i\xi_n)^{-1}$

   (b) $(|\xi|^2 + 1)^{-1}$

   (c) $(|\xi'|^2 + 1)^{-1}$

2. Let $x = (x', x'')$ belong to $\mathbb{R}^{n-d} \times \mathbb{R}^d$ with $d \in \{1, \dots, n-1\}$. For any $u \in C_0^\infty(\mathbb{R}^n)$, define

$$Vu(x') = \int_{\mathbb{R}^d} u(x', x'')\, \mathrm{d}x''.$$

   Write V as a Fourier integral operator and show that V can be extended to a continuous operator from $\mathcal{E}'(\mathbb{R}^n)$ to $\mathcal{E}'(\mathbb{R}^{n-d})$.

3. Let X and Y be open subsets of $\mathbb{R}^n$ and let $f : X \longrightarrow Y$ be a smooth diffeomorphism. Let $T : C_0^\infty(Y) \longrightarrow C^\infty(X)$ be defined by

$$Tu(x) = u(f(x)).$$

   Show that T is a Fourier integral operator.

4. Let X_1 and X_2 be two open subsets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Let ϕ_j belong to $C_0^\infty(X_j)$ and write $(\phi_1 \otimes \phi_2)(x_1, x_2) = \phi_1(x_1)\phi_2(x_2)$ for $x_j \in X_j$. Prove that if $u \in \mathcal{D}'(X_1 \times X_2)$ and $u(\phi_1 \otimes \phi_2) = 0$ for all $\phi_j \in C_0^\infty(X_j)$, then $u = 0$.

5. Let X_1 and X_2 be two open subsets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Consider $u_j \in \mathcal{D}'(X_j)$ and define $\varphi_2(x_1) = u_2(\phi(x_1, \cdot))$ and $\varphi_1(x_2) = u_1(\phi(\cdot, x_2))$ for a given $\phi \in C_0^\infty(X_1 \times X_2)$. Show that there exists $u \in \mathcal{D}'(X_1 \times X_2)$ satisfying

   (a) $u(\phi_1 \otimes \phi_2) = u_1(\phi_1)u_2(\phi_2)$ for all $\phi_j \in C_0^\infty(X_j)$,

   (b) and $u(\phi) = u_1(\varphi_2) = u_2(\varphi_1)$ for all $\phi \in C_0^\infty(X_1 \times X_2)$.

   Additionally, prove that if $u_j \in \mathcal{E}(X_j)$, (b) holds for $\phi \in C^\infty(X_1 \times X_2)$.

6. Let X_1 and X_2 be two open subsets of $\mathbb{R}_1^n$ and $\mathbb{R}_2^n$, respectively. Prove that every $K \in \mathcal{D}'(X_1 \times X_2)$, according to

$$\langle \mathcal{K}\phi, \psi \rangle = K(\psi \otimes \phi) \qquad \psi \in C_0^\infty(X_1),\ \phi \in C_0^\infty(X_2),$$

   defines a linear map $\mathcal{K}$ from $C_0^\infty(X_2)$ to $\mathcal{D}'(X_1)$ which is continuous in the sense that $\mathcal{K}\phi_j \to 0$ in $\mathcal{D}'(X_1)$ if $\phi_j \to 0$ in $C_0^\infty(X_2)$.

Comments.

(i) The distribution $u \in \mathcal{D}'(X_1 \times X_2)$ defined in Exercise 5 is called the tensor product of $u_1 \in \mathcal{D}'(X_1)$ and $u_2 \in \mathcal{D}'(X_2)$ and it is often denoted by $u = u_1 \otimes u_2$.

(ii) Exercise 6 is only part of the so-called Schwartz Kernel Theorem.