Linear algebra and matrices II
Department of mathematics and statistics
Autumn 2011
Exercise sheet 4

Exercises due date: Mon 28.11.2011 at 17.00
Corrections due date: Fri 2.12.2011 at 17.00

The main ideas of these exercises are

- Kernel and image of linear mappings
- Isoporphisms
- Eigenvalues
- Determinants

Exercise I

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(\bar{x}) = [7x_1 \quad x_1 + x_2 \quad 3x_2 - x_1]^T$.

1. Find the kernel $\mathrm{Ker}(L)$ of the mapping.
2. Is the mapping an injection?

Exercise II

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & -4 \end{bmatrix}$. Let us consider the linear mapping

$$L_A\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad L_A(\bar{x}) = A\bar{x}.$$

3. Find the kernel $\mathrm{Ker}(L_A)$ of the mapping.
4. What is the dimension of the kernel?
5. Find some generators for the image $\mathrm{Im}(L_A)$.
6. What is the dimension of the image $\mathrm{Im}(L_A)$?
7. Is the linear mapping L_A an injection, surjection or bijektion?
8. Basing on your previous results, decide whether A is invertible.

Exercise III

Let $L\colon V \to V'$ be a linear mapping.

9. Show that if the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} \subset V$ is not free,then the set

$$\{L(\bar{v}_1), L(\bar{v}_2), L(\bar{v}_3)\}.$$

is not free either.

Exercise IV

Let $A \in \mathbb{R}^{n\times n}$. The number $\lambda \in \mathbb{R}$ is an eigenvalue for the matrix if there is a non-zero vector $\bar{x} \in \mathbb{R}^n$ such that

$$A\bar{x} = \lambda\bar{x}.$$

The vector $\bar{x}$ satisfying the aforementioned condition is called an eigenvector relative to eigenvalue λ.

10. Show that $[1\ 1]^T$ is an eigenvector for the matrix

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

and find its correspondent eigenvalue.

11. Show that 5 is an eigenvalue for the matrix

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

and find all the eigenvectors relative to it.

Exercise V

Find out the matrix for the linear mapping $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ when

12. L rotates a vector 30° clockwise about the origin.

13. L gives the orthogonal projection of a vector onto the subspace span$\{\bar{w}\}$, where $\bar{w} = [3\ 1]^T$.

Let us consider again the matrices defined in the previous exercises.

14. Find geometrically, without using calculations, the eigenvectors for the matrix in exercise 12.

15. Find geometrically, without using calculations, the eigenvectors for the-matrix in exercise 13.

Exercise VI

Calculat $\det(A)$, when

16.
$$A = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$$

17.
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

18.
$$A = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 4 & 0 & -1 & -1 \\ -1 & 2 & -3 & -3 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

Information about determinants and handy methods to calculate them can be found following the link on the course's web page.

Exercise VII

19. Let $L\colon V \to V'$ be a linear mapping. Let us suppose that W is a subspace of V. Show that the image set $L[W] = \{L(\bar{w}) \mid \bar{w} \in W\}$ is a subspace of V'.

20. Express in your own words the result you just proved in the previous exercise. Do not use mathematical symbols.