Linear algebra and matrices II
Department of mathematics and statistics
Autumn 2011
Exercise sheet 3

Exercises due date: Mon 21.11.2011 at 16.00
Corrections due date: Fri 25.11.2011 at 17.00

The core ideas in these exercises are

- Definition of linear mapping
- Matrix of a linear mapping
- Kernel of a linear mapping

Exercise I

Find out whether f is linear when

1. $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$ for each $x \in \mathbb{R}$.
2. $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 5x$ for each $x \in \mathbb{R}$.
3. $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(\bar{x}) = [7x_1 \quad x_1 + x_2 \quad 3x_2 - x_1]^T$ for each $\bar{x} = [x_1\ x_2]^T \in \mathbb{R}^2$.
4. $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(\bar{x}) = [x_1 - 4 \quad 6x_2]^T$ for each $\bar{x} = [x_1\ x_2]^T \in \mathbb{R}^2$.
5. $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a mapping such that

$$f\Big([1\ 0]^T\Big) = [-2\,3]^T, \quad f\Big([0\,1]^T\Big) = [3\,1]^T \quad \text{ja} \quad f\Big([-3\,-2]^T\Big) = [0\,-12]^T.$$

Exercise II

Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping.

6. Let $L\Big([1\ 0]^T\Big) = [1\ 1]^T$ and $L\Big([0\ 1]^T\Big) = [-1\ 1]^T$. Find the image vectors $L\Big([1\ 4]^T\Big)$ and $L\Big([-2\ 3]^T\Big)$.
7. Let $L\Big([1\ 0]^T\Big) = [-2\ 0]^T$ and $L\Big([0\ 1]^T\Big) = [0\ 2]^T$. Find the image vector $L(\bar{x})$ of vector $\bar{x} \in \mathbb{R}^2$.

Exercise III

8. Show that the mapping $f\colon \mathbb{R} \to \mathbb{R}$ is linear if and only if there exists a real number a such that $f(x) = ax$ for each $x \in \mathbb{R}$. Write a your proof carefully and in a refined mathematical style.

Exercise IV

Let

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 1 & -5 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

From this matrix we get the linear mapping $L_A\colon \mathbb{R}^4 \to \mathbb{R}^3$ such that $L_A(\bar{x}) = A\bar{x}$ for each $\bar{x} \in \mathbb{R}^4$.

9. Find the images through L_A of the vectors $\{\bar{e}_1, \bar{e}_2, \bar{e}_4, \bar{e}_4\}$ of the natural basis. How can you spot them straight by looking at A?

Exercise V

Let $L\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. It can be proven that there exist a matrix $A \in \mathbb{R}^{n\times n}$ such that $L(\bar{x}) = A\bar{x}$ for each $\bar{x} \in \mathbb{R}^n$. The matrix $A$is called the matrix of the linear mapping L.

If A is the matrix of mapping L, the images of the vectors of the natural basis through L are just the columns of A. This information is useful upon finding the matrix of a linear mapping.

10. Find the matrix of linear mapping

$$L\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad f(\bar{x}) = [7x_1 \quad x_1 + x_2 \quad 3x_2 - x_1]^T$$

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Let $\bar{v} = [2 \ -1]^T$. Find the matrix of the linear mapping $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ in the following cases and draw a picture of vectors $\bar{v}$ and $L(\bar{v})$.

11. Mapping L stretches a vector three times as long and turns it into the opposite direction.
12. Mapping L returns the orthogonal projection of a vector onto subspace $\text{span}\{\bar{e}_2\}$, where $\bar{e}_2 = [0 \ 1]^T$.
13. Mapping L mirrors a vector with respect to line $y = x$.

Exercise VI

The kernel of a linear mapping $L\colon V \to W$ is the set

$$\text{Ker}(L) = \{\bar{v} \in V \mid L(\bar{v}) = \bar{0}\}.$$

In other words, it is the pre-image $f^{\leftarrow}\{\bar{0}\}$ of the set $\{\bar{0}\}$.

14. Find the kernel of mapping L_A in exercise IV.

15. Below is an attempt to prove that the kernel of a linear mapping is a subspace. However the proof is missing some details. Fix the proof using an appropriate mathematical fashion.

    Claim: Let $L\colon V \to W$ be a linear mapping. Then the set $\mathrm{Ker}(L)$ is a subspace of vector space V.

    Proof:

$$
\begin{aligned}
&1) \quad L(\bar{a}+\bar{b}) = L(\bar{a}) + L(\bar{b}) = \bar{0} + \bar{0} = \bar{0}\\
&2) \quad L(k\bar{a}) = kL(\bar{b}) = k \cdot \bar{0} = \bar{0}\\
&3) \quad L(\bar{0}) = \bar{0}
\end{aligned}
$$

Exercise VII

Let

$$P_2 = \{f\colon [0,1] \to \mathbb{R} \mid f \text{ is a polynomial function and } \deg f \leq 2\}.$$

We can define an addition and a scalar multiplication on the set P_2. If $f \in P_2$, $g \in P_2$ and $a \in \mathbb{R}$, then mappings $f+g$ and af can be defined as follows:

$$
\begin{aligned}
&f+g\colon\ [0,1] \to \mathbb{R}, \quad x \mapsto f(x)+g(x) \quad \text{and}\\
&af\colon\ [0,1] \to \mathbb{R}, \quad x \mapsto af(x).
\end{aligned}
$$

The set P_2, equipped with these operations, is a vector space (check page 34). For instance let us check the functions

$$f\colon [0,1] \to \mathbb{R},\ f(x) = x^2+1 \quad \text{and} \quad g\colon [0,1] \to \mathbb{R},\ g(x) = -4x.$$

Now functions $f+g$ and $3f$ look like this:

$$f+g\colon\ [0,1] \to \mathbb{R}, \quad x \mapsto x^2-4x+1 \quad \text{and} \quad 3f\colon\ [0,1] \to \mathbb{R} \quad x \mapsto 3x^2+3.$$

We can define an inner product on the space P_2 as

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

16. Find the norm $\|g\|$ of function g as defined above.
17. Let f and g be as above. calculate the projection $\mathrm{proj}_g f$.
18. Give two non-zero items of P_2 perpendicular to each other.