Linear Algebra and Matrix Theory I
Department of Mathematics and Statistics
Fall 2011
Exercises 2

Deadline for the exercises: Monday 19.9 18:00.
Deadline for the corrected exercises: Friday 23.9. 17:00.

The main contents of the exercises are:

- Inverse matrix
- Matrix calculation rules
- Solving systems of linear equations with Gaussian elimination method.

You don't have to write the assignments to your answer sheet.

Define $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$.

1. Suppose $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Calculate E_1A.
2. How is matrix A affected when multiplied with matrix E_1? An explanation is sufficient for an answer.
3. Suppose $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$. Calculate E_2A.
4. How is matrix A affected when multiplied with matrix E_2?
5. Suppose $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Calculate E_3A.
6. How is matrix A affected when multiplied with matrix E_3?

Just as a real number can have a inverse number, a matrix can also have a so called *inverse matrix* (def. 1.4.4 in the lecture material).

The number zero has no inverse number, thus it is impossible to divide with it (dividing is actually multiplying with an inverse number). For this reason, e.g. when solving equations, one must pay attention not to divide with zero.

Unlike real numbers, many matrices don't have an inverse matrix. Therefore one must be careful when handling them, as they can not be modified in the same fashion as real numbers.

7. Prove by using the definition of an inverse matrix that the inverse matrix of E_2 from exercise 3 is:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}.$$

8. Suppose that A and B are square matrices of same size and invertible. Show that the matrix AB is invertible and define its inverse matrix.

Suppose that A and B are square matrices of same size. Which of the following matrices in exercises 9–11 are identical with $(A-B)^2$? Remember to give an explanation or a counter example!

9. $A(A-B) - B(A-B)$

10. $A^2 - 2AB + B^2$

11. $A^2 - AB - BA + B^2$

The lecture material chapter 1.5 deals with unit triangular matrices and solving systems of linear equations. Get familiar with pages 15–22 through the following exercises.

The matrices in exercises 12–14 are one elementary row operation away from a unit upper triangular matrix. Which elementary row operation is required in each of the exercises? An explanation is sufficient for an answer.

12.
$$\begin{bmatrix} 1 & -5 & 4 & 0 & 1 \\ 0 & 0 & 2 & -8 & 3 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

13.
$$\begin{bmatrix} 1 & -5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

14.

$$\begin{bmatrix} 1 & -1 & 7 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

15. Define the augmented matrix of the following system of linear equations:

$$\begin{cases} \begin{array}{rcrcrcl} x_1 & & & + & 2x_3 & = & 1 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 2 \\ -x_1 & - & 2x_2 & & & = & 3 \end{array} \end{cases}$$

16. Reduce the matrix of the previous exercise into upper unit triangular form. You can ask advices from the instructors on how to write down the row operations.

17. What is the system of linear equations corresponding the upper unit triangular matrix formed in the previous exercise?

18. What can you tell about the relation between the two matrices of exercises 15 and 17?

In the exercises below, the augmented matrices of the systems of linear equations were reduced with Gaussian method into their current forms. What are the solutions of their systems of linear equations?

19.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

20.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

21.

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

22.

$$\left[\begin{array}{ccccc|c} 1 & 4 & 3 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

23. Use the Gaussian elimination method to solve the following system of linear equations:

$$\begin{cases} x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 4x_1 & + & 2x_2 & & & = & 8. \end{cases}$$

24. Equations that characterize chemical reactions must be balanced so that both sides of the equation hold equal amount of each atom. Define in the following exercise the coefficients x_1, x_2, x_3 and x_4 by finding the smallest possible natural numbers that satisfy the reaction:

$$x_1NO_2 + x_2H_2O \to x_3NO + x_4HNO_3.$$

25. Suppose that $a_{11}, a_{12}, a_{21}, a_{22}$ are real numbers. Assume also that c is a real number other than zero. Prove that the following equations

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{cases}$$

and

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ ca_{21}x_1 & + & ca_{22}x_2 & = & cb_2 \end{cases}$$

are equivalent.

Hint: Systems of linear equations are equivalent if they have identically same solutions. Assume first that $(x_1, x_2) = (r_1, r_2)$ is the solution of the first equation, and show that it is also solution of the second. And repeat this same reasoning vice versa.

Suppose that $A \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times 1}$ and $B \in \mathbb{R}^{m\times 1}$. Suppose that R is a solution of the matrix equation $AX = B$ and that S is the solution of the matrix equation $AX = \mathbf{0}$.

26. Show that $R + S$ is also a solution of the equation $AX = B$.

27. Suppose also that R' is a solution of the equation $AX = B$. Show that R' can be written in the form $R' = R + T$, where T is the solution of the equation $AX = \mathbf{0}$.

Hint: notice that $R' = R + (R' - R)$.

In the previous exercise you showed that the solutions of the equation $AX = B$ depend on the solutions of the equation $AX = \mathbf{0}$!