Linear Algebra and Matrix Theory I
Department of Mathematics and Statistics
Fall 2011
Exercises 1

Deadline for the exercises: Monday 12.9 18:00
Deadline for the corrected exercises: Friday 16.9. 17:00

You don't have to write the assignments to your answer sheet.

We start the course by going through some equation solving. An equation can usually be solved by deriving new equations from it that are much easier to solve than the original one. A new equation is derived so that all the solutions of the original equation are transmitted to the new one. However, not always do they have to have the same amount of solutions: the new equation might have more solutions than the original one. For this reason, in order to find out the solutions of the original equation, all solutions of the new equation must be checked whether they satisfy the original equation or not.

We say that two equations are *equivalent* if they have identically same solutions. In contrast to the previous notes of equation solving: an equation can also be solved by deriving equivalent equations of it that are easier to solve. Through this type of reasoning, the solutions of the original equation are identical to the solutions of the equivalent equation.

Finding suitable equivalent equations can be hard. Also, equation solving might turn out complicated and laborious if one needs to verify at every step that the successive equations are equivalent with each other. For this reason, the first method where the conclusive solutions are checked to satisfy the original equation is more safe and convenient. In general, checking your solutions in the end is a good way of avoiding careless mistakes!

Many equation solving methods are based on finding convenient equivalent equations. Later on in this course we will explore a method, which can be used to solve systems of linear equations.

Exercises 1–3 are not actually part of the theme of linear algebra, as not all of the equations are linear (check the lecture material chapter 1.1). However, these exercises enlighten equation solving and the concepts and phenomenons related to it.

1. Solve the equation $\sqrt{x+2} = -x$. (Hint: Raise both sides to power of 2. Remember to check the solutions that you get.)

2. Is $\sqrt{x+2} = -x$ equivalent with $x = -1$, i.e. do they have the same solutions?

3. Is $\sqrt{x+2} = -x$ equivalent with $x + 2 = x^2$?

4. Solve the system of linear equations

$$\begin{cases} -x + 3y = -2 \\ 2x + y = -3. \end{cases}$$

5. The equations of the previous exercise represented equations of lines. Draw a picture of these lines. How can you explain the solution with the picture?

6. Solve the system of linear equations

$$\begin{cases} -2x + 6y = -1 \\ x - 3y = 1. \end{cases}$$

7. The equations of the previous exercise represented equations of lines. Draw a picture of these lines. How can you explain the solution with the picture?

8. Solve the system of linear equations

$$\begin{cases} 2x - y = 3 \\ -4x + 2y = -6. \end{cases}$$

9. The equations of the previous exercise represented equations of lines. Draw a picture of these lines. How can you explain the solution with the picture?

In the lecture material chapter 1.2 we introduce charts of numbers that are called *matrices*. These charts are useful e.g. when solving systems of linear equations. Later on we will see that matrices also have other useful purposes. Through the following exercises we slowly start to construct the theory of matrices.

Matrices can be summed together and multiplied with each other. Many of the known operations that apply for real numbers also apply for matrices. Operations for matrices and transpose are defined in the lecture material chapter 1.2.

10. Define the coefficient matrix of the following system of linear equations:

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 3 \\ -x_1 + 5x_2 + x_3 = 4 \\ x_1 - 4x_2 + 2x_3 = -2. \end{cases}$$

Writing the result is sufficient for an answer.

Define

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 4 \end{bmatrix}.$$

In the exercises 11–17 calculate those matrices that are defined. If some of them are not, explain why.

11. $A + C$

12. $C - B$

13. $3A$

14. CA

15. AC

16. A^2

17. B^T

18. Assume that A is the coefficient matrix found in exercise 10. Define

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{ja} \quad B = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$$

Calculate the matrix product AX. How is $AX = B$ related to the equation presented in exercise 10?

19. Define

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \quad \text{ja} \quad B = \begin{bmatrix} -1 & 0 & 3 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 2 & -3 & -2 \\ 0 & -2 & 3 & 2 \end{bmatrix}.$$

Calculate the matrix product AB.

20. How does the matrix product BA look like when A and B are defined as in the previous exercise?

21. Give at least two different ways of how matrix multiplication differs from the product of real numbers.

22. Explain with your own words what a scalar matrix is. Do it without any mathematical symbols. (A scalar matrix is defined in the lecture material chapter 1.4.)

Find out in exercises 23–25, which of the claims are true and which are false. Remember to provide explanations or a counter example!

23. If A^2 is defined, then A is a square matrix.

24. If AB and BA are defined, then A and B are square matrices.

25. If $AB = B$ and B is not a null matrix, then $A = I$.

26. Suppose that $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times l}$ and let t be a real number. Show that $t(AB) = (tA)B$.

    Hint: Recall the condition of when two matrices are same and examine arbitrary elements of the matrices. Remember to start the proof by listing your assumptions. The proof could start e.g.: "Suppose that $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times l}$."