Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 11
28.11-02.12.2011

1. Let $n > 1$ and suppose $f\colon S^{n-1} \to S^{n-1}$ is a continuous mapping. Write $S^n = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x|^2 + |t|^2 = 1\}$ and define $\Sigma f\colon S^n \to S^n$ by the formula
$$\Sigma f(x,t) = \begin{cases} (|x| \cdot f(x/|x|), t), & \text{if } x \neq 0, \\ (x,t), & \text{if } x = 0. \end{cases}$$
Prove that Σf is continuous.

2. Suppose $f\colon S^n \to S^n$ is **even**, i.e. $f(x) = f(-x)$ for all $x \in S^n$. Prove that $\deg f$ is even integer and if n is even then $\deg f = 0$. (Hint: f factors through the projective space $\mathbb{R}P^n$).
For every $m \in \mathbb{Z}$ give an example of an even mapping $f\colon S^1 \to S^1$ with $\deg f = 2m$.

3. a) For every $x \in \overline{B}^n, x \neq 0$ let
$$\alpha(x) = 2\sqrt{\frac{1-|x|}{|x|}}.$$
Define $h\colon \overline{B}^n \to S^n$ by
$$h(x) = \begin{cases} (\alpha(x)x_1, \alpha(x)x_2, \dots, \alpha(x)x_n, 1-2|x|), & \text{if } x \neq 0 \\ e_{n+1} = (0,\dots,1), & \text{if } x = 0. \end{cases}$$
Prove that h is a well-defined continuous surjective mapping which restriction to B^n is a homeomorphism to $S^n \setminus \{-e_{n+1}\}$ and which maps S^{n-1} onto $-e_{n+1}$. Deduce that h induces a homeomorphism $\overline{B}^n/S^{n-1} \cong S^n$.
b) Define $f\colon S^n \to S^n$ so that $f|B_+ = h \circ g$, where g is a standard homeomorphism $B_+ \to \overline{B}^n$, $g(x_1,\dots,x_n,x_{n+1}) = (x_1,\dots,x_n)$ and $f|B_-$ is a constant mapping that maps everything to the south pole $-e_{n+1}$.
Prove that f is a well-defined continuous mapping and $f(x) \neq -x$ for all $x \in S^n$. Deduce that $\deg f = 1$.

4. Suppose (X,A) is a topological pair and A is a closed subset of X. Let $f\colon A \to Y$ and let $p\colon X \sqcup Y \to X \cup_f Y$ be the canonical quotient projection. Then $p|X \setminus A$ is an open injection and $p|Y$ is a closed injection. In particular both restriction are embeddings, $p(X \setminus A)$ is open in $X \cup_f Y$ and $p(Y)$ is closed in $X \cup_f Y$.

5. Suppose Z is obtained from Y by attaching n-cells. Show that the set of open cells depends only on the pair (Z,Y) (Hint: consider components of $X \setminus Y$). Assuming Z is Hausdorff show that the same is true for closed cells.

6. Suppose $p\colon X \to Y$ is a quotient mapping and $A \subset Y$ is open or closed. Show that $p|p^{-1}A\colon p^{-1}A \to A$ is a quotient mapping.

7. a) Suppose Z is obtained Y by attaching n-cells and C is a compact subset of Z. Then Z intersects only finitely many open cells of Z.
b) Suppose X is a CW-complex and C is a compact subset of Z. Then there exists $n \in \mathbb{N}$ such that $C \subset X^n$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.