Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 8
Solutions

1. Let X be a non-empty set. Define $C_n(X)$ to be the free abelian group generated on the set X^{n+1} for $n \geq 0$ and $C_n(X) = 0$ for $n < 0$. Prove that the definition
$$\partial(x_0, \dots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_n)$$
defines a boundary operator that makes the collection $C(X) = \{C_n(X), \partial\}$ a chain complex. Prove that $C(X)$ has an augmentation $\varepsilon\colon C_0(X) \to \mathbb{Z}$ defined by $\varepsilon(x) = 1$ on generators.

   For a fixed $x \in X$ and every $n \geq 0$ define homomorphism $x\colon C_n(X) \to C_{n+1}(X)$ by
$$x(x_0, \dots, x_n) = (x, x_0, \dots, x_n).$$
Prove that
$$(\partial_{n+1} x + x\partial_n)(y) = \begin{cases} y, & \text{if } n \neq 0, \\ y - \varepsilon(y)x, & \text{if } n = 0. \end{cases}$$
for all $y \in C(X)$. Deduce that the complex $\widetilde{C}(X)$ is acyclic.

   Solution: The fact that $C(X)$ is a chain complex, i.e. $\partial \circ \partial = 0$ is proved completely analogically to the proof singular chain complex of a topological space is a complex, so we skip the details.

   The fact that is augmentation is also easy - it is clearly surjective (since X is non-empty) and
$$\varepsilon\partial_1(x_0, x_1) = \varepsilon(x_1 - x_0) = 1 - 1 = 0.$$
   Let $n > 0$ and $y = (x_0, \dots, x_n)$ is a free generator. Then
$$(\partial x + x\partial)(y) = \partial(x, x_0, \dots, x_n) + \sum_{i=0}^{n} (-1)^i (x, x_0, \dots, \hat{x}_i, \dots, x_n) =$$
$$= (x_0, \dots, x_n) + \sum_{i=1}^{n+1} (-1)^i (x_0, \dots, \widehat{x_{i-1}}, \dots, x_n) + \sum_{i=0}^{n} (-1)^i (x, x_0, \dots, \hat{x}_i, \dots, x_n).$$
   Change of variables in the last sum shows that all terms cancel out, except for the first one, so
$$(\partial x + x\partial)(y) = y.$$
   Since this is true for all the generators, it is true for all elements.
   We are left with the case $n = 0$. In that case
$$(\partial x + x\partial)(x_0) = \partial(x, x_0) + 0 = x_0 - x = x_0 - \varepsilon(x_0)x.$$

Since this is true for all generators, this must be true for all points.

In particular if we restrict x to $\tilde{C}$, then $x\tilde{C} \to \tilde{C}$ is a chain homotopy from identity mapping of $\tilde{C}$ to zero mapping. Since chain homotopic mappings induce same mappings in homology, it follows that $\mathrm{id}\colon H_n(\tilde{C}) \to H_n(\tilde{C})$ is a zero mapping for all $n \in \mathbb{N}$, which can only be possible if $H_n(\tilde{C})$ is a trivial group for all $n \in \mathbb{N}$, so $\tilde{C}$ is acyclic.

2. Suppose C, D are chain complexes and $f_n, g_n\colon C_n \to D_n$ homomorphisms defined for every $n \in \mathbb{Z}$. Suppose for every $n \in \mathbb{N}$ there exists a homomorphism $H_n\colon C_n \to D_{n+1}$ with the property

$$\partial_{n+1}H_n + H_{n-1}\partial_n = f_n - g_n \text{ for all } n \in \mathbb{Z}.$$

Prove that $f - g = \{f_n - g_n | n \in \mathbb{Z}\}$ is a chain mapping.
Deduce that if g is a chain mapping, also f is. In other words **mapping that is homotopic to a chain mapping is a chain mapping itself**.
Solution: Denote $h = f - g$. Then

$$\partial H + H\partial = h.$$

Straight calculation shows that

$$\partial h = \partial\partial H + \partial H\partial = \partial H\partial,$$

$$h\partial = \partial H\partial + H\partial\partial = \partial H\partial,$$

so $\partial h = h\partial$, i.e. h is a chain mapping.

Suppose g is a chain mapping. Then $f = (f - g) + g = h + g$ is a chain mapping, as a sum of two chain mappings.

3. Define a homotopy $H_n\colon C_n(X) \to C_{n+1}X$ by

$$H_n(\sigma) = \sigma_\sharp(H_n(\Delta_n)),$$

where $H_n(\Delta_n)$ is the image of $\mathrm{id}\colon \Delta_n \to \Delta_n$ under $H_n\colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n) \subset C_n(\Delta_n)$. Prove (using the corresponding property of $H_n\colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n)$) that H is a chain homotopy between id and barycentric subdivision operator $S\colon C(X) \to C(X)$.

Solution: Let us first show that $H\colon LC(D) \to LC(D)$ is **natural** with respect to affine mappings. Put precisely that D and D' be two convex subsets of some finite-dimensional vector spaces $\alpha\colon D \to D'$ is an affine mapping. Then *alpha* induces homomorphism $\alpha_\sharp\colon LC(D) \to LC(D')$, by restriction of $\alpha_\sharp\colon C(D) \to C(D')$. This is well defined, since if $\beta\colon \Delta_n \to D$ is affine, then $\alpha_\sharp(\beta) = \alpha \circ \beta\colon \Delta_n \to D'$ is affine.
We claim that $H \circ \alpha_\sharp = \alpha_\sharp H$. This is shown by induction on n:
$H_0 = 0$, so the claim is trivially true for $n = 0$. Suppose claim is proved for $n - 1 \geq 0$. Then

$$\alpha_\sharp H_n(f) = \alpha_\sharp(b_f(f - H_{n-1}(\partial f))) = b_{\alpha\circ f}\alpha_\sharp(f - H_{n-1}(\partial f))) =$$

$$= b_{\alpha\circ f}(\alpha_\sharp(f) - H_{n-1}(\alpha_\sharp\partial f)) = b_{\alpha\circ f}(\alpha_\sharp(f) - H_{n-1}(\partial\alpha_\sharp(f))) = H_n(\alpha_\sharp(f)).$$

Here we used the facts that $\alpha_\sharp$ is a chain mapping i.e. commutes with boundary, the inductive assumption on H_{n-1} and easy observation that

$$\alpha_\sharp b_f = b_{\alpha_\sharp(f)}\alpha_\sharp.$$

This concludes the proof of commutative relation $H \circ \alpha_\sharp = \alpha_\sharp H$.

Now let $\sigma\colon \Delta_n \to X$ be a singular n-simplex in X. We need to show that

$$(\partial H_n + H_{n-1}\partial)(\sigma) = \sigma - S(\sigma).$$

By definition we have

$$H_n(\sigma) = \sigma_\sharp(H_n(id\colon \Delta_n \to \Delta_n)),$$

hence also

$$H_{n-1}\partial(\sigma) = \sum_{i=0}^{n}(-1)^i(\partial^i\sigma)_\sharp(H_{n-1}((id\colon \Delta_{n-1} \to \Delta_{n-1}))).$$

Now $\partial^i\sigma = \sigma\colon \varepsilon^i$, where $\varepsilon^i\colon \Delta^{n-1} \to \Delta^n$ is an affine mapping. Also

$$(\partial^i\sigma)_\sharp = (\sigma\colon \varepsilon^i)_\sharp = \sigma_\sharp \circ (\varepsilon^i)_\sharp.$$

By naturality of H_{n-1} with respect to affine mappings we have that

$$(\varepsilon^i)_\sharp H_{n-1}(\mathrm{id}_{n-1}) = H_{n-1}((\varepsilon^i)_\sharp)(\mathrm{id})) = H_{n-1}(\varepsilon^i).$$

Hence

$$(\partial H_n+H_{n-1}\partial)(\sigma) = \sigma_\sharp(\partial H_n(\mathrm{id}_n))+H_{n-1}(\sum_{i=0}^{n}(-1)^i\varepsilon^i) = \sigma_\sharp(\partial H_n(\mathrm{id}_n))+H_{n-1}(\partial(\mathrm{id}_n)).$$

We know from lecture notes that

$$\partial H_n(\mathrm{id}_n)) + H_{n-1}(\partial(\mathrm{id}_n) = \mathrm{id}_n - S(\mathrm{id}_n).$$

Plugging it into equation above gives

$$(\partial H_n + H_{n-1}\partial)(\sigma) = \sigma_\sharp(\mathrm{id}_n - S(\mathrm{id}_n)) = \sigma - S(\sigma)$$

by the definition of S.

4. Let

$$B_+ = \{x \in S^n \mid x_{n+1} \geq 0\} \text{ and}$$
$$B_- = \{x \in S^n \mid x_{n+1} \leq 0\}.$$

Use homology and excision axioms to show that the inclusions $i\colon (B_+, S^{n-1}) \to (S^n, B_-)$ and $j\colon (B_-, S^{n-1}) \to (S^n, B_+)$ induce isomorphism in relative homology (for all dimensions).

Solution: Let $U = S^n \setminus \{-e_{n+1}\}$. Then U is open subset of S^n and the inclusion of pairs $(B_+, S^{n-1} \hookrightarrow (U, B_-/\setminus\{-e_{n+1}\})$ is a homotopy equivalence. Hence it induces isomorphisms in relative homology for all $n \in \mathbb{N}$. Since $A = \{-e_{n+1}\}$ is a closed set which is contained in the interior $\{x \in S^n \mid x_{n+1} < 0\}$ of B_-, excision property implies that the inclusion $(U, B_-/\setminus\{-e_{n+1}\}) \hookrightarrow (S^n, B_-)$ induces isomorphisms in homology. Hence the composite $i\colon (B_+, S^{n-1}) \to (S^n, B_-)$ also induces isomorphisms in relative homology (for all dimensions). The claim for j is proved similarly.

5. a) Suppose $U \subset \mathbb{R}^n$ is open and $x \in U$. Prove that

$$j_*\colon H_m(U, U \setminus \{x\}) \cong H_m(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$

for all $m \in \mathbb{N}$. Here j is an obvious inclusion of pairs.
b) Suppose $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are both open and non-empty and there is a homeomorphism $f\colon U \to V$. Prove that $n = m$.(Hint: remove a point)
Solution: a) Let $A = \mathbb{R}^n \setminus U$, $V = \mathbb{R}^n \setminus \{x\}$. Then $\overline{A} = A \subset \operatorname{int} V = V$, so excision property implies that inclusion induces isomorphism $H_m(\mathbb{R}^n \setminus A, V \setminus A) \cong H_m(\mathbb{R}^n, V)$ for all $m \in \mathbb{N}$. But this is precisely the claim.

b) Let $x \in U$. Homeomorphism f defines homeomorphism of pairs $(U, U \setminus \{x\} \to (V, V \setminus \{f(x)\}$, hence $H_n(U, U \setminus \{x\}) \cong H_n(V, V \setminus \{f(x)\})$.
By a) we obtain that $\mathbb{Z} = H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$. If $m \neq n$, then $H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) = 0$. Hence we must have $m = n$.

6. Suppose $f\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism. Show that f maps interior B^n onto itself and the boundary S^{n-1} also onto itself.
Solution: It is enough to show that if $x \in B^n$ then also $f(x) \in B^n$. Assume contrary - $f(x) \in S^{n-1}$. Then f induces homeomorhism between $X = \overline{B}^n \setminus \{x\}$ and $Y = \overline{B}^n \setminus \{f(x)\}$. But X has the same homotopy type as S^{n-1}, in particular $n-1$-dimensional reduced homology group of X is non-trivial. Y, on the other hand, is convex (linear homotopy to origin suffice), in particular its reduced homology groups are all trivial. Contradictions follows.

7. Show that $U = S \setminus \{e_{n+1}\}$ is homeomorphic to $\mathbb{R}^n$ via stereographic projection through the north pole e_{n+1}.
Stereographic projection of the point $y \in U$ is defined to be the unique point in $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ which lies on the line spanned by y and e_{n+1}. Construct the explicit formula for the stereographic projection and its inverse.
Solution: The line L_y that goes through y and e_{n+1} has parametric representation

$$ty + (1-t)e_{n+1}, t \in \mathbb{R}.$$

It follows that a point $z(t) = ty + (1-t)e_{n+1} = (ty_1, \dots, ty_n, ty_{n+1} + 1 - t)$ lies on this line and belongs to $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$ if and only if

$$t(y_{n+1} - 1) + 1 = ty_{n+1} + 1 - t = 0$$

i.e. if and only if

$$t = \frac{1}{1 - y_{n+1}}.$$

Hence for the stereographic projection $f\colon U \to \mathbb{R}^n$ we obtain formula

$$f(y_1, \dots, y_n, y_{n+1}) = \frac{1}{1 - y_{n+1}}(y_1, \dots, y_n).$$

This is well-defined, since $y_{n+1} \neq 1$ for $y \in U$ and is clearly continuous.

To construct formula for the inverse we take a point $x \in \mathbb{R}^n \subset \mathbb{R}^{n+1}$, the line L_x spanned by x and e_{n+1} and try to find a unique point in U that lies on L_x. Now the representation for L_x is

$$tx + (1-t)e_{n+1}, t \in \mathbb{R}.$$

It follows that a point $z(t) = tx + (1-t)e_{n+1} = (tx_1, \ldots, tx_n, 1-t) \in L_x$ is in the set U if and only if $t \neq 0$ and

$$t^2(|x|^2 + 1) - 2t + 1 = t^2(x_1^2 + \ldots + tx_n^2) + (1-t)^2 = |z(t)|^2 = 1 \text{ i.e.}$$

$$t(|x|^2 + 1) = 2.$$

Hence for the inverse g of f we obtain formula

$$g(x) = \frac{2}{|x|^2 + 1} x + \frac{|x|^2 - 1}{|x|^2 + 1} e_{n+1}.$$

Clearly g defined by this formula is continuous. From construction it follows that g and f are inverses of each others. If one wants, one can also check formally from the formulas that $g = f^{-1}$.