Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 7
Solutions

1. a) Suppose X is a non-empty space and $x \in X$. For every path-component X_a of X which does not contain x choose a point $y_a \in X_a$. Prove that the set

$$\{[y_a - x] \mid a \in \mathcal{A}\}$$

is a basis for $\tilde{H}_0(X)$, which is thus a free abelian group.
Here $\mathcal{A}$ is a set of all path-components of X that do not contain x.

b) Suppose $X = S^0 = \{1, -1\}$ is a 2-point discrete space. Show that $\tilde{H}_0(X) \cong \mathbb{Z}$ with $1 - (-1)$ a generator and $\tilde{H}_n(X) = 0$ for $n \neq 0$.

Solution: a) By Corollary 3.1.3 and Proposition 3.1.4 $H_0(X)$ is a free abelian group with basis

$$\{[y_a] \mid a \in \mathcal{A}\} \cup \{[x]\}.$$

Now $\tilde{H}_0(X) = \operatorname{Ker} \varepsilon_*$. First of all

$$\varepsilon_*[y_a - x] = \varepsilon(y_a) - \varepsilon(x) = 0,$$

so $[y_a - x] \in \tilde{H}_0(X)$. Suppose

$$a = \sum_{i=1}^{n} k_i[y_a] + k[x] \in \operatorname{Ker} \varepsilon_*,$$

then $\varepsilon_*(a) = \sum_{i=1}^{n} k_i + k = 0$, so $k = -\sum k_i$, hence

$$a = \sum_{i=1}^{n} k_i([y_a] - [x]) = \sum_{i=1}^{n} k_i([y_a - x]).$$

Thus the set $\{[y_a - x] \mid a \in \mathcal{A}\}$ generates the group $\tilde{H}_0(X)$. It remains to show that it is free. Suppose

$$0 = \sum_{i=1}^{n} k_i([y_a - x]) = \sum_{i=1}^{n} k_i[y_a] + k[x],$$

where $k = -\sum_{i=1}^{n} k_i$. Since the set $\{[y_a] \mid a \in \mathcal{A}\} \cup \{[x]\}$ is free, it follows that $k_1 = \ldots = k_n = k = 0$.

b) The claim about $\tilde{H}_0(X)$ follows from a). Also for $n \neq 0$

$$\tilde{H}_n(X) = H_n(X) = H_n(\{-1\} \oplus H_n(1) = 0 \oplus 0 = 0.$$

2. Prove that Mobius band has the same homotopy type as S^1.

Solution: We think of Mobius band as a quotient space $X = I^2/\sim$, where $I = [0, 1]$ and $(0, t) \sim (1, 1 - t)$ for all $t \in I$. Consider the subspace

$$Y = \{[(x, 1/2)]\}$$

of X. Then Y is homeomorphic to S^1, so it enough to show that the inlusion $i\colon Y \hookrightarrow X$ is a homotopy equivalence. Let us define $q\colon X \to Y$ by

$$q([x, y]) = [x, 1/2] \text{ and}$$

$H\colon X \times I \to X$ by

$$H([x, t], t') = [x, (1 - t')t + t'/2].$$

Then $i \circ q(a) = H(a, 1)$, $a = H(a, 0)$ for all $a \in X$ and

$$\begin{aligned} H([0, t], t') &= [0, (1 - t')t + t'/2] = [1, 1 - (1 - t')t - t'/2] = \\ &= [1, 1 - t' - t + t't + t'/2] = [1, (1 - t')(1 - t) + t'/2] = \\ &= H([1, 1 - t], t'). \end{aligned}$$

Hence H is well defined. Consider the commutative diagram

$$\begin{array}{ccc} I^2 \times I & & \\ \downarrow{\scriptstyle \pi\times\mathrm{id}} & \searrow^{\tilde{H}} & \\ X \times I & \xrightarrow{H} & X. \end{array}$$

Here $\pi\colon I^2 \to X$ is a canonical projection and $\tilde{H}$ is defined by the formula

$$\tilde{H}((x, t), t') = (x, (1 - t')t + t'/2).$$

Now $I^2 \times I$ is compact and X is Hausdorff, so $\pi \times \mathrm{id}$ is a quotient mapping (Topology II). Since $\tilde{H}$ is continuous, it follows that H is continuous.
Thus H is a homotopy from identity mapping to $i \circ q$. Clearly $q \circ i = \mathrm{id}$. We have shown that i is a homotopy equivalence.

3. a) Suppose Y is a contractible space and X is any space. Suppose $f\colon X \to Y$ and $g\colon Y \to X$ are continuous mappings. Prove that both f and g are homotopic to constant mappings. Also prove that Y is path-connected.

   b) Suppose Y is a non-empty space. Prove that the following conditions are equivalent:
   1) Y is contractible.
   2) The set $[X, Y]$ is a singleton for any space X.
   3) Y is path-connected and the set $[Y, X]$ is a singleton for every non-empty path-connected space X.
   4) Y has a homotopy type of a singleton space.

Solution: a) Since Y is contractible there exists $y \in Y$ such that $\mathrm{id} \approx c_y$, where c_y is a constant mapping, $c_y(x) = y$ for all $x \in Y$. Then

$$f = \mathrm{id} \circ f \approx c_y \circ f = c_y,$$

$$g = g \circ \mathrm{id} \approx g \circ c_y = c_{g(y)}.$$

Let $H\colon Y \times Y \to Y$ be a homotopy $\mathrm{id} \approx c_y$. Then for any $x \in X$ the path $\alpha_x\colon I \to Y$ defined by

$$\alpha_x(t) = H(x, t)$$

is a path from x to y in Y. In particular Y is path-connected.

b) Suppose Y is contractible. As above we see that there exists $y \in Y$ such that every $f\colon X \to Y$ is homotopic to a constant mapping $c_y\colon x \mapsto y$. Hence

$[X, Y]$ is a singleton. Also Y is path-connected. Suppose X is path-connected. As above we see that every mapping $g\colon Y \to X$ is homotopic to a constant mapping c_x for some $x \in X$. Let $x, x' \in X$. Then there exists a path α from x to x' and the mapping $H\colon Y \times I \to Y$ defined by

$$H(y,t) = \alpha(t)$$

is a homotopy $c_x \approx c_{x'}$. Hence all constant mappings $Y \to X$ are homotopic, so all mappings $Y \to X$ are homotopic.
Consider singleton space $\{y\}$ and let $i\colon \{y\} \to Y$ be inclusion, $q\colon Y \to \{y\}$ the unique mapping. Then $q \circ i = \mathrm{id}$ and $i \circ q = c_y$ is homotopic to identity. Hence i is a homotopy equivalence.
We have shown that 1)⇒2), 1)⇒3), 1)⇒4).

Suppose 2) or 3). Then in particular $[Y, Y]$ is a singleton, so $\mathrm{id}\colon Y \to Y$ is homotopic to any constant mapping in Y. Hence Y is contractible.

Suppose 4). There is y such that the only possible mapping $q\colon Y \to \{y\}$ is a homotopy equivalence. Let $i\colon \{y\} \to Y$ be homotopy inverse of q. Suppose X is a space and $f\colon X \to Y$ is a mapping. Then

$$f = \mathrm{id} \circ f \approx i \circ q \circ f = c_{i(y)},$$

hence $[X, Y]$ is a singleton. In other words 4) ⇒ 2).

4. a) Suppose $f\colon (X, A) \to (Y, B)$ is a mapping of pairs. Suppose that $f\colon X \to Y$ as well as $f|A\colon A \to B$ are homotopy equivalences. Prove that

$$f_*\colon H_n(X, A) \to H_n(Y, B)$$

is an isomorphism.

b) Let

$$X = \bigcup_{n \in \mathbb{N}_+} \{1/n\} \times I \cup \{0\} \times I \cup I \times \{0\}$$

(so-called "topological comb space ") and $x_0 = (0, 1)$. Prove that a constant mapping $f\colon (X, x_0) \to (x_0, x_0)$ is such that its restrictions to $X \to x_0$ and $x_0 \to x_0$ are homotopy equivalences, but f is not a homotopy equivalence (as a mapping of pairs).

Solution: a) By Corollary 3.2.5 $f_*\colon H_n(X) \to H_n(Y)$ as well as $(f|A)_*\colon H_n(A) \to H_n(B)$ are isomorphisms for every $n \in \mathbb{Z}$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X,A) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) \\
\downarrow{\scriptstyle f|_* \cong} & & \downarrow{\scriptstyle f_* \cong} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f|_* \cong} & & \downarrow{\scriptstyle f_* \cong} \\
H_n(B) & \xrightarrow{i_*} & H_n(Y) & \xrightarrow{j_*} & H_n(Y,B) & \xrightarrow{\partial} & H_{n-1}(B) & \longrightarrow & H_{n-1}(Y)
\end{array}$$

with exact rows. Five Lemma (Lemma 2.2.9) implies the claim.

b) mapping $\{x_0\} \to \{x_0\}$ is a homeomorphism, in particular a homotopy equivalence. Let $i\colon \{x\} \to X$ be an inclusion. Then $f \circ i = \mathrm{id}$. Let

$H_1, H_2, H_3 \colon X \times I \to X$ be homotopies defined by

$$H_1((x,t),t') = (x,(1-t')t),$$

$$H_2((x,t),t') = ((1-t')x, 0).$$

$$H_3((x,t),t') = (0, 1-t').$$

Then H_1 is a homotopy id $\approx pr_1$, where $pr_1(x,y) = (x,0)$, H_2 is a homotopy $pr_1 \approx c$, where $c \colon X \to X$ is a constant mapping $(x,y) \mapsto (0,0)$ and H_3 is a homotopy $c \approx i \circ f$. Hence $i \circ f \approx \mathrm{id}$.

Suppose $f \colon (X, x_0) \to (x_0, x_0)$ is a homotopy equivalence of pairs, then its inverse must be the only mapping $i \colon (x_0, x_0) \to (X, x_0)$, $i(x_0) = x_0 \in X$. Now $i \circ f$ is homotopic to identity as a mapping of pairs i.e. there exists a homotopy $H \colon X \to I \to X$ such that

$$H(x,0) = x,$$

$$H(x,1) = x_0$$

for all $x \in X$ and $H(x_0, t) = x_0$ for all $t \in I$.
Let

$$U = \{(x,y) \in X \mid y > 0\} \subset X.$$

Then U is open and $H(\{x_0\} \times I) \subset U$, hence

$$\{x_0\} \times I \subset H^{-1}(U),$$

which is then open in $X \times I$, by continuity.
Since both $\{x_0\}$ and I are compact, there exist open neighbourhood V of x_0 in X such that

$$\{x_0\} \times I \subset V \times I \subset H^{-1}(U)$$

(see Topology II). In other words $H(V \times I) \subset U$. Since V is a neighbourhood of $x_0 = (0,1)$, there exists $n \in \mathbb{N}$ such that $x_n = (1/n, 1) \in V$. Hence mapping $\alpha \colon I \to X$ defined by

$$\alpha(t) = H(x_n, t)$$

is a path from x_n to x_0 in U. But this is impossible since x_n and x_0 clearly belong to different components of U.

5. Suppose K is a **finite** Δ-complex. For every geometric n-simplex σ of K choose a point $x_\sigma \in \operatorname{int} \sigma$ and let $U = |K^n| \setminus \{x_\sigma | \sigma \in K_n/\sim\}$. Prove that U is open in K^n and the inclusion $|K^{n-1}| \hookrightarrow U$ is a homotopy equivalence.
Deduce that the inclusion $i \colon (|K^n|, |K^{n-1}|) \to (|K^n|, U)$ induces isomorphisms in relative homology in all dimensions.

Solution: Suppose σ is a geometrical simplex in $|K^n|$. Then $U \cap \sigma$ is either σ (if $\dim \sigma < n$) or $\sigma / setminus\{x_\sigma\}$ (if $\dim \sigma = n$), so is open in σ in every case. Since $|K^n|$ has weak topology coherent with simplices, it follows that U is open in $|K^n|$.
Let

$$Z = \sqcup_{\sigma \in K^n} \sigma,$$

$$\tilde{U} = Z \setminus \{x_\sigma | \sigma \in K_n/\sim\}$$

and let $\pi\colon Z \to |K^n|$ be a canonical projection (which is quotient mapping with respect to the weak topology on $|K^n|$). Then $\tilde{U} = \pi^{-1}U$ is open in Z. We define $\tilde{H}\colon \tilde{U} \times I \to |K^n|$ so that $\tilde{H}(x,t) = x$ for $x \in |K^{n-1}|$ and

$$H(x,t) = (1-t)x + tx/|x|$$

for $x \in \sigma, x \neq x_\sigma$, where σ is an n-dimensional simplex of K. Here we identify σ with $\overline{B}^n$ via a homeomorphism which maps x_σ to 0 (Proposition 1.1.10). Then there is a (unique) mapping $H\colon U \times I \to |K^n|$ such that the diagram

$$\begin{array}{ccc} \tilde{U} \times I & & \\ \Big\downarrow{\scriptstyle \pi|\times \mathrm{id}} & \searrow^{\tilde{H}} & \\ U \times I & \xrightarrow{H} & |K^n|. \end{array}$$

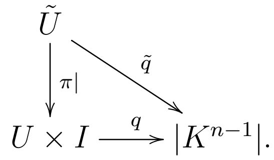
commutes. To show H is continuous we need to prove that $\pi| \times \mathrm{id}$ is a quotient mapping. Since K is finite $Z \times I$ is compact, so continuous surjective $\pi \times \mathrm{id}\colon Z \times I \to |K^n| \times I$ is a closed mapping, hence quotient mapping, provided we know that $|K^n|$ is Hausdorff. Let us go back to that later. Also notice that for us "K is finite"means that K has finitely many **geometrical** simplices, but the set of simplices K can be infinite, in which case Z is not compact. However in this case we can always reduce the amount of simpices in K to finite, without altering the amount of geometrical simplices.
Next we use the following well-known topological result (proof of which is left to the reader, in case he/she is not familiar with it):
Suppose $f\colon X \to Y$ is quotient mapping and $U \subset Y$ is open (or closed). Then the restriction $f|\colon f^{-1}U \to U$ is also a quotient mapping.

Hence in the end we obtain that $\pi| \times \mathrm{id}$ is a quotient mapping, which suffices to assure H is continuous. Clearly H is a homotopy from identity to the mapping $q = H(\cdot, 1)\colon U \to |K^{n-1}|$, or, to be precise to the mapping $i \circ q$, where i is the inclusion $|K^{n-1}| \hookrightarrow U$. Also q is constructed so that $q \times i = \mathrm{id}$. Hence q is a homotopy equivalence of i.

Consider inclusion of pairs $i\colon (|K^n|, |K^{n-1}|) \to (|K^n|, U)$. Then the restriction $i|\colon |K^{n-1}| \to U$ is the inclusion, which we just proved to be homotopy equivalence. Also the restriction $i\colon |K^n| \to |K^n|$ is a homotopy equivalence, since it is just identity mapping.
Now the last claim follows from exercise 4a).

The only problem left is that we did not verify that $|K^n|$ is Hausdorff. This actually follows from more general results on CW-complexes we will prove later in the course.
This can also be proved directly by induction on n. For $n = 0$ this is clear, since $|K^0|$ is discrete. Suppose $x, y \in |K^n|, x \neq y$ and the claim is true for $n-1$. Suppose y is in the interior of an n-simplex σ, which we identify with the open disk B^n, so that y corresponds to 0. Then no matter where x is, there is small enough $r > 0$ so that open disk $V = B(0,r)$ of radius r does not contain x, while $W = |K^n| \setminus \overline{B}(x,r)$ does contain x, in which case V and W are disjoint neighbourhoods of y and x. By symmetry this

also handles the case x is in the interior of some n simplex. We are left with the case $x, y \in |K^{n-1}|$. We take $U = |K^n| \setminus \{x_\sigma | \sigma \in K_n/\sim\}$ as above and $\tilde{U} = p^{-1}(U)$. By the general topological results mentioned above we know that the restriction $p|\colon \tilde{U} \to U$ is a quotient mapping. We define $\tilde{q}\colon \tilde{U} \to |K^{n-1}|$ as above, so that it is identity on simplices of dimension smaller than n and then a natural restriction to the boundary on $\sigma/ \setminus \{x_\sigma\}$ on every n-simplex σ. As above we easily seen that this mapping quotiens out in the diagram



giving us a continuous mapping $q\colon U \to |K^{n-1}|$, which is, in fact, **a retraction** of U onto $|K^{n-1}|$.
Now since U is open in $|K^n|$, it is enough to prove that x and y have disjoint neighbourhoods V, W in U. But by inductive assumption they have disjoint neighbourhoods V', W' in $|K^{n-1}|$ and we just assert

$$V = q^{-1}(V'), W = q^{-1}(W').$$

Remark: The only technical problem we faced was to show that $\pi \times \mathrm{id}\colon X \times I \to Y \times I$ is a quotient mapping, when $\pi\colon X \to Y$ is, and that is why we had to restrict ourselves to the finite case. This is not necessary - it is always true that $\pi \times \mathrm{id}\colon X \times I \to Y \times I$ is a quotient mapping, when $\pi\colon X \to Y$ is, but the proof is not trivial so we skip it in this course. You can find it in Maunder, Algebraic Topology (Theorem 6.2.4).

6. Suppose C', C, D, D' are chain complexes, $f, g, h\colon C \to D$, $k, m\colon D \to D'$, $l\colon C' \to C$ are chain mappings.
a) Suppose H is chain homotopy from f to g, H' chain homotopy from g to h. Prove that $H + H'$ is a chain homotopy from f to h. Deduce that the relation "f and g are chain homotopic"is an equivalence relation in the set of all chain mappings $C \to D$.
b) Prove that $k \circ H$ is a chain homotopy from $k \circ f$ to $k \circ g$ and $H \circ l$ is a chain homotopy from $f \circ l$ to $g \circ l$.
c) Suppose H'' is a chain homotopy from k to m. Then $H'' \circ f + m \circ H$ and $k \circ H + H'' \circ g$ are chain homotopies from $k \circ f$ to $m \circ g$.
Solution: a) We have equations

$$\partial H + H\partial = g - f,$$

$$\partial H' + H'\partial = h - g.$$

Adding them together gives

$$\partial(H + H') + (H + H')\partial = (g - f) + (h - g) = h - f,$$

so $H + H'$ is a chain homotopy from f to h. This implies that the relation "f and g are chain homotopic"is transitive. It is also reflexive, since 0 is a chain homotopy from f to f, for every chain mapping f, and it is symmetric since if H is a chain homotopy from f to g, $-H$ is a chain homotopy from g to f.

b) Again we begin with equation

$$\partial H + H\partial = g - f.$$

Applying k on the left gives us

$$k\partial H + kH\partial = kg - kf.$$

But k is a chain mapping, so $k\partial = \partial k$, thus we obtain

$$\partial(kH) + (kH)\partial = kg - kf,$$

which implies that kH is a chain homotopy from kf to kg.
The second claim is proved in the similar way.

c) This is combination of a)and b) - by b) $H'' \circ f$ is a homotopy from $k \circ f$ to $m \circ f$, while $m \circ H$ is a homotopy from $m \circ f$ to $m \circ g$. Hence by a) $H'' \circ f + m \circ H$ is a chain homotopy from $k \circ f$ to $m \circ g$. The other claim is proved similarly.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.