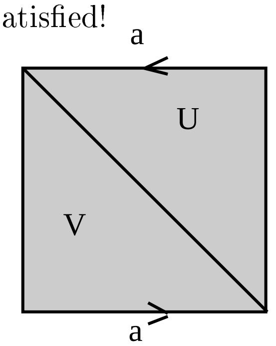
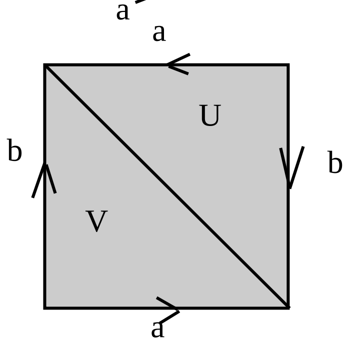
Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 3
26.09-30.09.2011

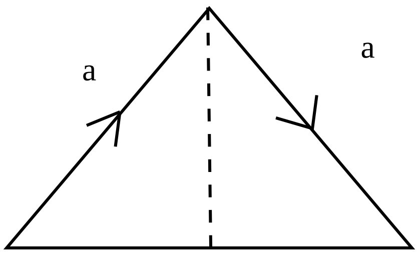
1. Give a Δ-complex structures with 2 triangles to the Mobius band and to the projective plane, according to the picture below. Remember to specify the ordering of the vertices and check that the requirements of the definition are satisfied!





2. Suppose K is a Δ-complex and σ is an n-simplex of K.
Show that the restriction of the characteristic mapping $f_\sigma | \operatorname{int} \Delta_n$ to the interior of Δ_n is a homeomorphism to its image and $|K|$ is a disjoint union of the sets $\{f_\sigma(\operatorname{int} \Delta_n)\}$ (meaning that two sets are either the same or disjoint). Prove that the topology of $|K|$ is co-induced by the set of charachteristic mappings $\{f_\sigma\}_{\sigma \in K}$.

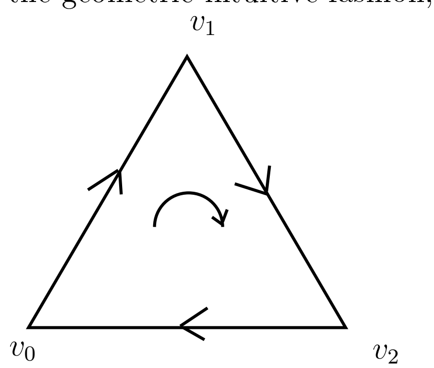
3. Suppose in an ordered triangle $[v_0, v_1, v_2]$ i.e. 2-simplex you identify two faces $[v_0, v_1]$ and $[v_1, v_2]$ (preserving the ordering, as usual). What familiar space is this quotient space homeomorphic with? (Hint: cut the triangle in half, as the picture indicates, making it a simplicial complex made up by two triangles, then do the identification, then glue triangles back. Drawing pictures might help! Remember to keep the track of the ordering.)**the picture and more exercises on the next page!**



What if we identify sides $[v_0, v_1]$ and $[v_0, v_2]$ instead?

4. Suppose X is a quotient space of the cylinder $S^1 \times I$, with identifications $(x, 0) \sim (-x, 0)$, $x \in S^1$.
a) Define a Δ-complex structure on X. (Hint: start with a square). b) Prove that X is homeomorphic to the Mobius strip. (Hint: cut, rearrange and glue again - just as in the previous exercise.

5. Consider the triangle with "clockwise orientation"(defined on the boundary in the geometric-intuitive fashion, as in the picture below).



Go through all the permutations of the set $\{0, 1, 2\}$ and show that even permutations preserve the clockwise orientation, while odd permutations switch it to the counter-clockwise orientation.

6. Consider an ordered tetrahedron with vertices $v_0, \ldots, v_3$. Draw all the pairs of 2 dimensional faces and make sure that 0-face and 2-face have the same geometric orientation, 1-face and 3-face also have the same geometric orientation, while in all the other cases orientation is different (one clockwise, the other counter-clockwise).

7. Suppose K is a Δ-complex. Define its first barycentric subdivision $K^{(1)}$ (as a Δ-complex) by miming the definition for the simplicial complexes.
a) Show that $K^{(1)}$ is a Δ-complex L that has the following property:
1)Every geometrical n-simplex of L has exactly $n + 1$ vertices i.e. no vertices of the same simplex are identified.
Show that if a Δ-complex L has the property 1), all the characteristic mappings of all simplex are bijective.
b) Suppose L is a Δ-complex that has property 1). Prove that its barycentric subdivision $L^{(1)}$ has the property of a simplicial complex - the intersection of two geometric simplices is either empty or a common face of both simplices.

Conclude that every polyhedron of some Δ-complex is a polyhedron of some simplicial complex.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.