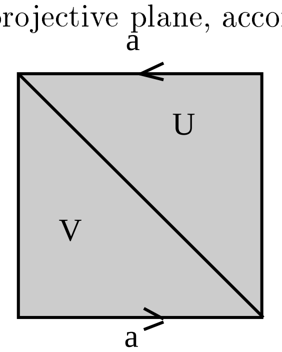
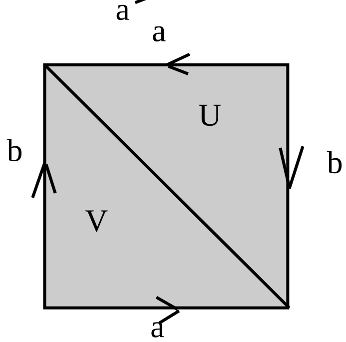
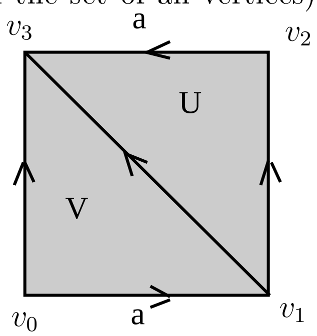
Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 3
Solutions

1. Give a $\Delta$-complex structures with 2 triangles to the Mobius band and to the projective plane, according to the picture below.



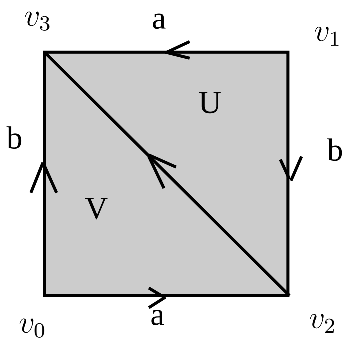




**Solution:** It is enough to order vertices in a linear manner (i.e. define a linear order in the set of all vertices). One possible choice is given in the pictures




below.

Here we have ordered 2-simplices $V = [v_0, v_1, v_3], U = [v_1, v_2, v_3]$ and all their faces. The edges $[v_0, v_1]$ and $[v_2, v_3]$ are identified, as well as the diagonal of the square which represents faces $[v_1, v_3]$.
This also forces identifications $v_0 \sim v_2$ and $v_1 \sim v_3$ on the vertices. There are no other identifications.




Here we have ordered 2-simplices $V = [v_0, v_2, v_3], U = [v_1, v_2, v_3]$ and all their faces. There are following identifications on edges :
$[v_0, v_2] \sim [v_1, v_3]$,
$[v_0, v_3] \sim [v_1, v_2]$,
and of course the common diagonal $[v_2, v_3]$.

These identifications also force identifications $v_0 \sim v_1$ and $v_2 \sim v_3$ on the vertices. There are no other identifications.

2. Suppose $K$ is a $\Delta$-complex and $\sigma$ is an $n$-simplex of $K$.
Show that the restriction of the characteristic mapping $f_\sigma | \operatorname{int} \Delta_n$ to the interior of $\Delta_n$ is a homeomorphism to its image and $|K|$ is a disjoint union of the sets $\{f_\sigma(\operatorname{int} \Delta_n)\}$ (meaning that two sets are either the same or disjoint). Prove that the topology of $|K|$ is co-induced by the set of characteristic mappings $\{f_\sigma\}_{\sigma \in K}$.

**Solution:** Let us start with the last claim. Suppose $U \subset |K|$ and let $\pi\colon Z \to |K|$ be a canonical quotient mapping. Here

$$Z = \bigsqcup_{\sigma \in K} \sigma.$$

Then by the definition $U$ is open in $|K|$ if and only if $\pi^{-1}U$ is open in $Z$. Since $Z$ has a topology of disjoint union, $\pi^{-1}U$ is open in $Z$ if and only if $\pi^{-1}U \cap \sigma$ is open in $\sigma$ for every $\sigma \in K$. Let $i\colon \sigma \to Z$ be a canonical inclusion. Then

$$\pi^{-1}U \cap \sigma = i^{-1}\pi^{-1}U.$$

Let $\alpha\colon \Delta_n \to \sigma$ be a unique order-preserving simplicial mapping, where $n = \dim \sigma$. Since $\alpha$ is a homeomorphism, $i^{-1}\pi^{-1}U$ is open in $\sigma$ if and only if $\alpha^{-1}i^{-1}\pi^{-1}U = f_\sigma^{-1}U$ is open in $\Delta_n$.

Hence $U$ is open in $|K|$ if and only if $f_\sigma^{-1}U$ is open in $\Delta_n$ for every $\sigma \in K$. This means precisely that the topology of $|K|$ is co-induced by the collection of characteristic mappings $\{f_\sigma\}_{\sigma \in K}$.

Next we prove that every characteristic mapping is closed. Since $\Delta_n$ is compact, this would be clear if we would knew that $|K|$ is Hausdorff. The polyhedron is always Hausdorff, but it requires a proof, so we won't go into that, and instead we will prove the claim directly. Suppose $C \subset \Delta_n$ is closed. Let $\sigma$ be an $n$-simplex and $\sigma'$ be an arbitrary $m$-simplex of $K$. Let $J$ be the set of all "common faces"of $\sigma$ and $\sigma'$ i.e. faces of $\sigma$ which in $K$ are identified with some face of $\sigma$. For every $\sigma_0 \in J$ let $\sigma_0'$ be a face of $\sigma$ such that $\sigma_0 \sim \sigma_0'$. Then

$$f_{\sigma'}^{-1}(f_\sigma(C)) = \bigcup_{\sigma_0 \in J} \beta^{-1}(C \cap \sigma_0'),$$

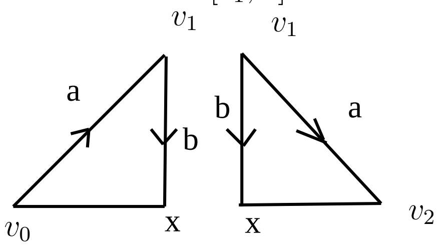
where $\beta\colon \sigma_0 \to \sigma_0'$ is a unique simplicial homeomorphism that preserves order of vertices. Since $J$ is finite and $\beta^{-1}C \cap \sigma_0'$ is closed in $\sigma_0$, hence in $\sigma'$, we conclude that $f_{\sigma'}^{-1}(f_\sigma(C))$ is closed in $C$. By the claim which is already prove above, $f_\sigma(C)$ is closed in $|K|$. Hence $f_\sigma\colon \Delta_n \to |K|$ is closed.

Next we use the following topological fact (prove it, if its not familiar): suppose $f\colon X \to Y$ is a closed mapping and $A \subset Y$. Then the restriction $f|\colon f^{-1}(A)\colon A$ is a closed mapping.

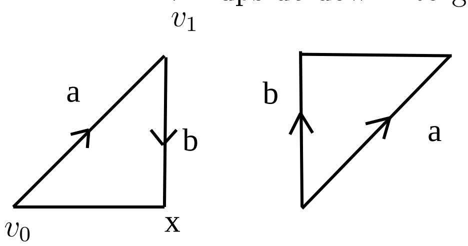
We apply this fact to $f\colon \Delta_n \to |K|$ and $A = f_\sigma(\operatorname{int} \Delta_n)$, hence obtaining that the restriction of $f$ to $f_\sigma^{-1}(A) = \operatorname{int} \Delta_n$ is closed mapping to its image $f_\sigma(\operatorname{int} \Delta_n)$. It remains to prove that it it injection. But this follows from the fact that all non-trivial identifications inside a simplex can happen only on the boundary.

For the similar reason two simplices that have "common points"(in sence of identifications in $K$) in their interiors must be identified, so their characteristic mappings are the same. This implies that interiors are disjoint. Clearly every point of $|K|$ belongs to some interior - just choose the smallest (in sense of dimension) simplex which image contains that point.

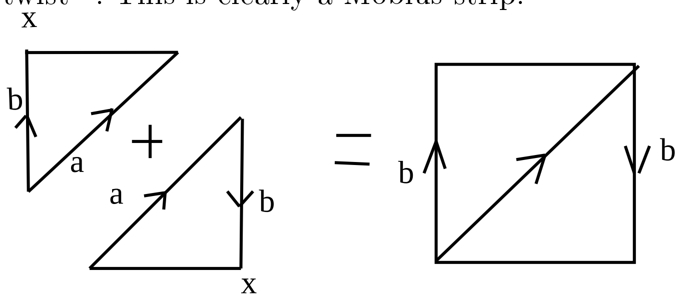
3. Suppose in an ordered triangle $[v_0, v_1, v_2]$ i.e. 2-simplex you identify two faces $[v_0, v_1]$ and $[v_1, v_2]$ (preserving the ordering, as usual). What familiar space is this quotient space homeomorphic with?
**Solution:** Cut the triangle into two triangles along the line between $v_1$ and the midpoint $x$ of $[v_0, v_2]$. This amounts to representing the space as a polyhedron of a $\Delta$-complex generated by two 2-simplices - $U = [v_0, v_1, x]$ and $V = [v_1, x, v_2]$, with sides $a = [v_0, v_1] \sim [v_1, v_2]$ identified, as well as the common side $b = [v_1, x]$ also identified.



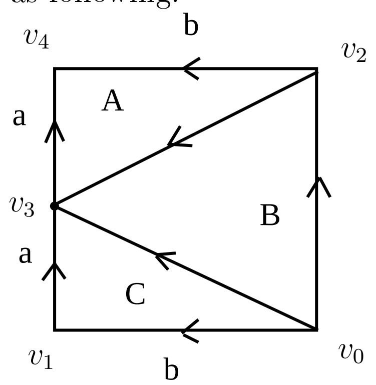

Next rotate $V$ "upside down "to get the following situation:



We see that if we now glue $V$ to $U$ along the original identification $[v_0, v_1] \sim [v_1, v_2]$ we obtain a familiar looking square with vertical sides identified "with a twist ". This is clearly a Mobius strip.

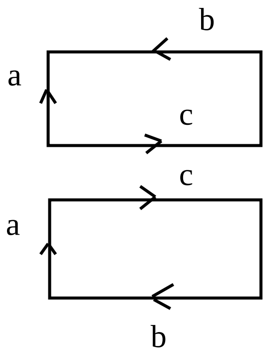



4. Suppose $X$ is a quotient space of the cylinder $S^1 \times I$, with identifications $(x, 0) \sim (-x, 0)$, $x \in S^1$.
   a) Define a $\Delta$-complex structure on $X$. b) Prove that $X$ is homeomorphic to the Mobius strip.
   **Solution:** a) The cylinder $S^1 \times I$ can be obtained from the square $I^2 = [0, 1]^2$ by identifying points $(x, 0)$ and $(x, 1)$ for all $x \in I$. The extra identification $(x, 0) \sim (-x, 0)$, $x \in S^1$ then corresponds to identifications $(0, y) \sim (0, y + 1/2)$, $0 \leq y \leq 1/2$. Hence we can represent $X$ as a polyhedron of a $\Delta$-complex $X$ as following:

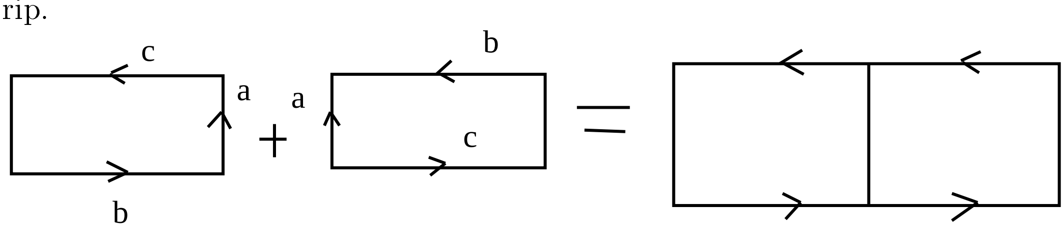



The order of vertices is given by their indices. $K$ has 3 2-simplices $A = [v_2, v_3, v_4], B = [v_0, v_2, v_3], C = [v_0, v_1, v_3]$ and all their faces. Except for the obvious "common faces"obtained from the triangulation of a square we also have identifications $[v_0, v_1] \sim [v_2, v_4]$ (corresponding geometrical simplex is denoted $b$ in the picture) and $[v_1, v_3] \sim [v_3, v_4]$ (corresponding geometrical simplex is denoted $a$ in the picture) on edges. These also force identifications $v_0 \sim v_2$ and $v_1 \sim v_3 \sim v_4$ on vertices. There are no other identifications.

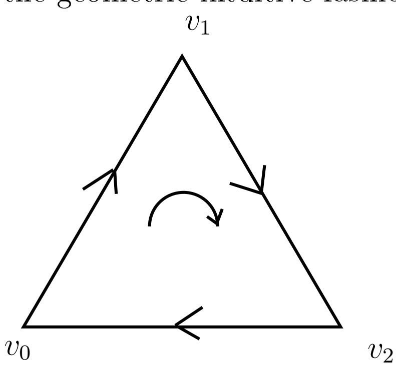
b) Cut the square along the horizontal line $y = 1/2$ (which is denoted $c$ in the picture below) to obtain two rectangles.




Now rotate the one below to get its "mirror image", then glue rectangles along the edge $a$. As the picture clearly suggests one ends up with the Mobius strip.

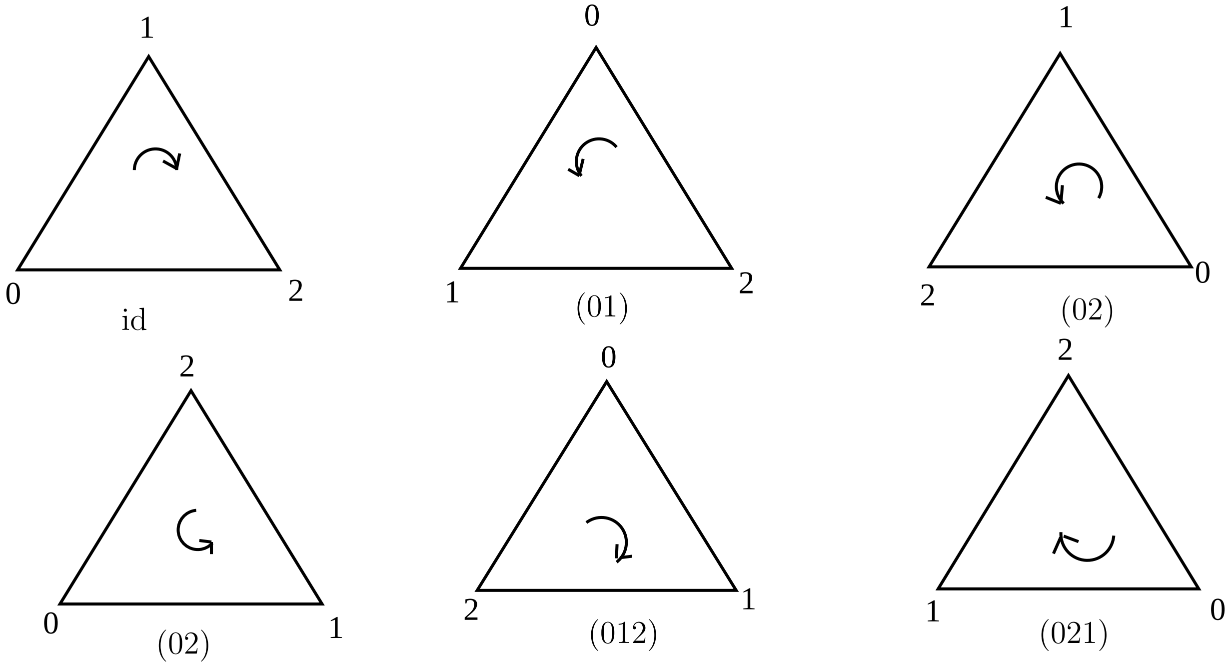



5. Consider the triangle with "clockwise orientation"(defined on the boundary in the geometric-intuitive fashion, as in the picture below).



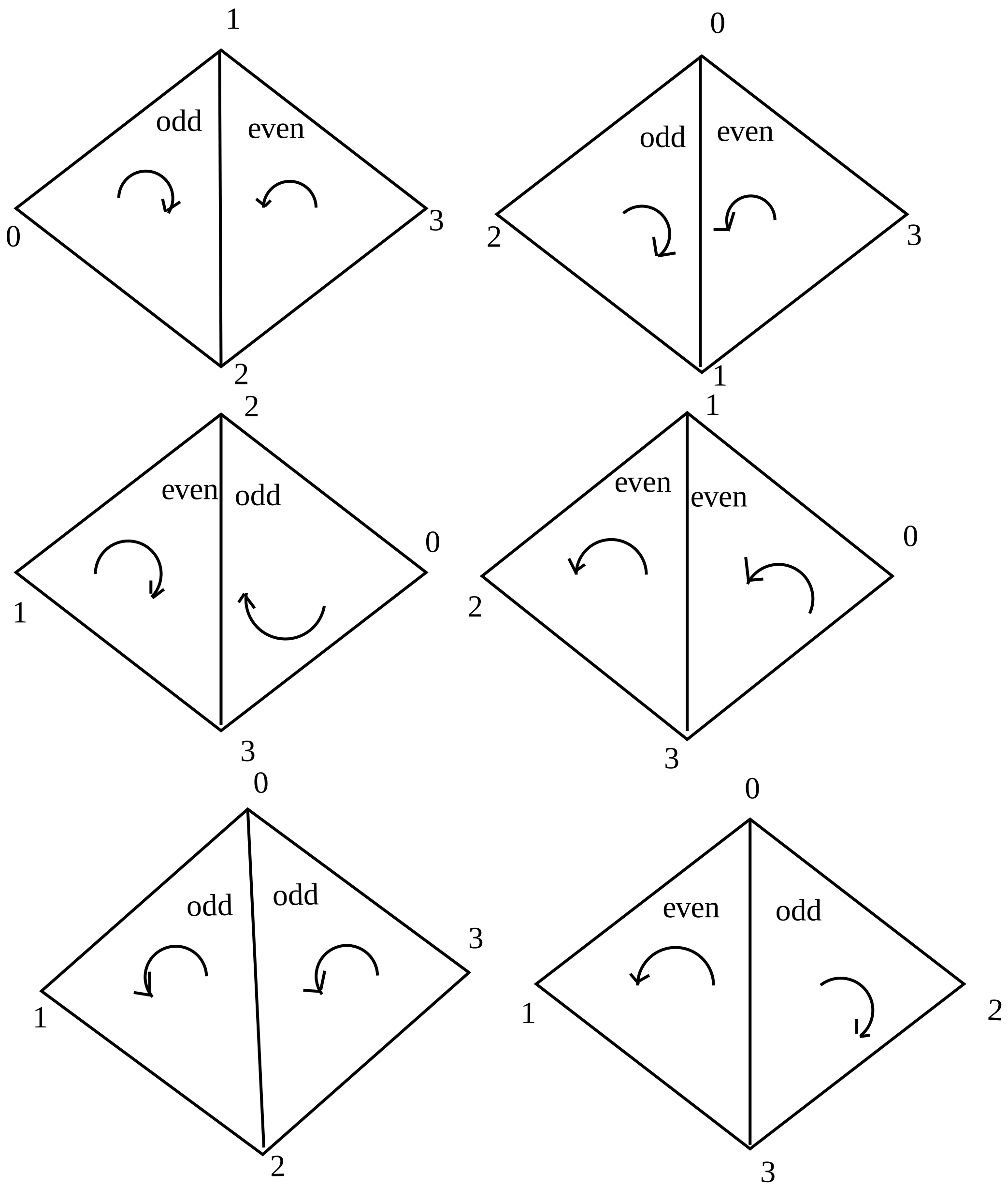

Go through all the permutations of the set $\{0,1,2\}$ and show that even permutations preserve the clockwise orientation, while odd permutations switch it to the counter-clockwise orientation.

**Solution:** There are 6 permutations. We use the notation $(a_0a_1\dots a_n)$ for a cyclic permutation that takes $a_i$ to $a_{i+1}$ (mod $n$) and leaves the other elements fixed.




Since id, (012) and (021) are even and the other permutations are odd, we see that the claim is true.

6. Consider an ordered tetrahedron with vertices $v_0, \dots, v_3$. Draw all the pairs of 2 dimensional faces and make sure that 0-face and 2-face have the same geometric orientation, 1-face and 3-face also have the same geometric orientation, while in all the other cases orientation is different (one clockwise, the other counter-clockwise).




**Solution:**

7. Suppose $K$ is a $\Delta$-complex. Define its first barycentric subdivision $K^{(1)}$ (as a $\Delta$-complex) by miming the definition for the simplicial complexes.
a) Show that $K^{(1)}$ is a $\Delta$-complex $L$ that has the following property:
1)Every geometrical $n$-simplex of $L$ has exactly $n+1$ vertices i.e. no vertices of the same simplex are identified.
Show that if a $\Delta$-complex $L$ has the property 1), all the characteristic mappings of all simplex are bijective.
b) Suppose $L$ is a $\Delta$-complex that has property 1). Prove that its barycentric subdivision $L^{(1)}$ has the property of a simplicial complex - the intersection of two geometric simplices is either empty or a common face of both simplices.

Conclude that every polyhedron of some $\Delta$-complex is a polyhedron of some simplicial complex.

**Solution:** The definition of the barycentric subdivision is completely analogical to the simplicial case - we define $K'$ to be the set of all simplices of

the form $\operatorname{conv}\{b(\sigma_0), \ldots, b(\sigma_n)\}$, where $\sigma_0 < \sigma_1 < \ldots < \sigma_n \in K$. Since every simplicial mapping takes a barycentre to the barycentre, it is easy to see that identifications on $K$ define identifications in $K'$ in a natural way. More precisely $\operatorname{conv}\{b(\sigma_0), \ldots, b(\sigma_n)\} \sim \operatorname{conv}\{b(\sigma'_0), \ldots, b(\sigma'_n)\}$ if $\sigma_i \sim \sigma'_i$ for all $i = 0, \ldots, n$.
Clearly $|K'|$ is homeomorphic to $|K|$ in a natural way.

Now if $\sigma_0 < \sigma_1 < \ldots < \sigma_n \in K$, then $b(\sigma_n) \in \int \sigma_n$, while $b(\sigma_i) \in \partial\sigma_n$ for all $i = 0, \ldots, n-1$. Hence $b(\sigma_n) \neq b(\sigma_i$ for $i \neq n$. Continuing by induction we see that all vertices of a simplex in $K'$ are different in $|K'|$.

Suppose $L$ is a $\Delta$-complex with property 1). Let $\sigma$ be an $n$-simplex of $L$. If $f_\sigma$ is not injection, then some different faces of $\sigma$ are identified. This implies immediately that corresponding vertices are identified, in particular $\sigma$ has at most $n$ vertices, which is a contradiction with property 1).

Hence every geometrical simplex in $|L|$ "looks like "a simplex. But still two different simplices can have more that one common side, that is why $L$ is not necessarily a simplicial complex yet.

Consider simplices $\sigma = \operatorname{conv}\{b(\sigma_0), \ldots, b(\sigma_n)\}, \sigma' = \operatorname{conv}\{b(\sigma'_0), \ldots, b(\sigma'_m)\}$ in $L'$, where $\sigma_0 < \sigma_1 < \ldots < \sigma_n \in L$, $\sigma'_0 < \sigma'_1 < \ldots < \sigma'_m \in L'$. Suppose $\sigma_n$ and $\sigma_m$ are identified. Then, after the identifications, $\sigma$ and $\sigma'$ are two simplices in the first barycentric subdivision of a simplex, which is a simplicial complex. In particular intersection of two simplices is a common face (or empty).

Next suppose $\sigma_n$ and $\sigma_m$ are not identified. Then their intersection is a subset of their boundaries. Now the intersection $\sigma \cap \partial\sigma_n$ is a simplex in a barycentric subdivision of $\sigma_{n-1}$ and $\sigma' \cap \partial\sigma'_m$ is a simplex in a barycentric subdivision of $\sigma'_{m-1}$. Now by induction we may assume that the claim is true for these simplices. Since the claim is clear in case on of the simplices has dimension 0, we are done.

To be precise we have proved that $L'$ has the property of a simplicial complex: intersection of two simplices in $|L|$ is a common side (or empty). Formally it is still a collection of simplices which may lie in different vector spaces.
Let $A$ be the set of all geometric vertices of $|L'|$. Construct a vector space $V = \mathbb{R}^{(A)}$ with $A$ as a basis and make a copy of $L'$ in $V$ in an obvious way. Then this copy will be a simplicial complex, whose polyhedron is homeomorphic to the polyhedron of $L'$ in an obvious way.