Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 2
19.09-23.09.2011

1. Suppose V is a vector space. Show that the collection $K = \{\sigma_i\}_{i \in I}$ of simplices in V is a simplicial complex if and only if
   1) For every simplex σ in K its every face also belongs to K.
   2') For every $x \in \bigcup_{i \in I} \sigma_i$ there is a unique $i \in I$ such that x is an interior point of the simplex σ_i.

2. Suppose L is a subcomplex of a simplicial complex K. Show that
   a) The weak topology on the simplicial complex $|L|$ is the same as the relative topology on $|L|$ induced by the weak topology of $|K|$.
   b) $|L|$ is closed in $|K|$.

3. a) Suppose σ is a simplex in $\mathbb{R}^m$, with vertices $\{v_0, \dots, v_n\}$. Prove that

$$\operatorname{diam} \sigma = \max\{|v_i - v_j|\},$$

   where $|\cdot|$ is a standard norm on $\mathbb{R}^m$.
   b) Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Let σ' be a simplex in a first barycentric division $K^{(1)}$, with vertices $\{b(\sigma_0), b(\sigma_1), \dots, b(\sigma_n)\}$, where $\sigma_0 < \dots < \sigma_n = \sigma \in K$. Prove that

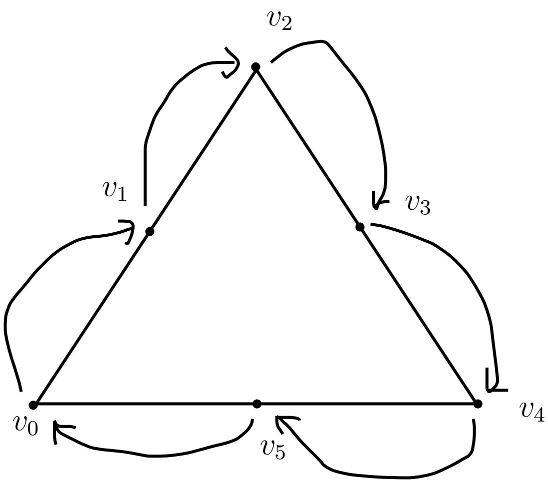
$$\operatorname{diam} \sigma' \leq \frac{n}{n+1} \operatorname{diam} \sigma$$

4. Suppose g is a simplicial approximation of the continuous mapping $f\colon |K| \to |K'|$. Show that

$$f(\operatorname{St}(v)) \subset \operatorname{St}(g(v))$$

   for every vertex $v \in K$.

5. Consider the boundary of the equilateral triangle σ as a 2-simplex with vertices v_0, v_2, v_4. For odd $i = 1, \dots 5$ denote by v_i the barycentre of the 1-simplex $[v_{i-1}, v_{i+1}]$, where we identify $v_6 = v_0$.
   Let $K = K(\partial\sigma)$. Let $f\colon |K| \to |K|$ be the unique simplicial mapping $f\colon |K^{(1)}| \to |K^{(1)}|$ defined by $f(v_i) = v_{i+1}$. Prove that as a mapping $f\colon |K| \to |K|$ f does not have a simplicial approximation, but as a mapping $f\colon |K^{(1)}| \to |K|$ f has exactly 8 simplicial appoximations. List all approximations.
   (see the picture and the rest of the exercises on the other side!)



6. a) Suppose $m \in \mathbb{N}$. Let K be an m-dimensional simplicial complex and K' be a simplicial complex whose dimension is $> m$. Show that every continuous mapping $f\colon |K| \to |K'|$ is homotopic to a mapping, which is not surjective (Hint: simplicial approximation).
   b) Suppose $m < n$. Prove that any continuous mapping $f\colon S^m \to S^n$ is homotopic to a constant mapping.

7. Suppose $x \in |K|$.
   a)Define $L = \{\sigma \in K | x \notin \sigma\}$. Show that L is a simplicial complex and
   $$|K| \setminus |L| = \mathrm{St}(x).$$
   Conclude that $\mathrm{St}(x)$ is an open neighbourhood of x in $|K|$.
   b)Suppose $x \in |K|$ and all the vertices of $\mathrm{car}(x)$ are $v_0, \ldots, v_n$.
   Prove that
   $$\mathrm{St}(x) = \bigcup\{\mathrm{int}\,\sigma \mid \mathrm{car}(x) < \sigma\} = \bigcup\{\mathrm{int}\,\sigma \mid v_0, \ldots, v_n \text{ are vertices of } \sigma\}.$$
   and
   $$\mathrm{St}(x) = \bigcap_{i=0}^{n} \mathrm{St}(v_i).$$

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.