Foreword

Mathematics based on the abstract set-theoretic approach as we know it today is quite a modern science - it is at most something like 200 years old. The development of the "foundations" of mathematics started in the 19th century with the work of scientists like Cauchy, Riemann and Cantor, to name only a few.

Pretty soon top mathematicians discovered a number of basic topological questions that sounded very elementary and had a clear intuitive " answer ", but the actual precise mathematical proof that this answer is right seemed very difficult.

For instance consider Euclidean vector spaces $\mathbb{R}^n$, which provide an important enviroment for analysis, linear algebra and topology. It seems intuitively clear that different spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, $m \neq n$, should not be homeomorphic as spaces, but it took many years of research and the development of advanced mathematical tools and techniques to actually prove this result, known as " The invariance of domain " principle. One of the reasons this claim seems obvious is our intuition regarding the notion of " dimension " - it seems that, for example, the plane $\mathbb{R}^2$ has "more space" than the real line $\mathbb{R}$ and the 3-dimensional " space " $\mathbb{R}^3$ has even more "filling", so it seems impossible to even fill bigger dimensional space with smaller dimensional. However, some 20 years before the invariance of domain was actually proven by Brouwer, the Italian mathematician Giuseppe Peano managed to constuct a surjective continuous mapping $f\colon I \to I^2$ (also known as " a space-filling curve "), thus showing that you can actually " fill " a bigger dimensional object with a smaller dimensional. In light of this discovery some mathematicians even doubted whether the invariance of domain was true at all.

Soon the scientists that had tried to solve these topological problems realized that the right strategy lies in the construction of **invariants** i.e. "objects" that are associated with spaces and mappings between spaces and somehow reflect their properties. If these invariants are in some sence " simpler" than the studied space i.e. reflect only some of its properties, they are easier to handle, so the difficult problem might turn into simpler problem, defined for these invariants, which is possible to solve. If a certian inavariant fails to offer enough help, some other invariant or a combination of invariants might do the trick, so it is also important to develop "enough" of these invariants. Today most of these invariants are algebraic in nature, that is why the field in now known as **algebraic** topology. In the beginning, for instance in Brouwer's days, these invariants were merely discrete objects, for example integer numbers, that is why then this approach was known as **combinatorial** topology.

The construction of one such invariant (or to be precise rather the family of invariants)- **the singular homology theory** (with integer coefficients) is the main subject and the goal of this course. We will prove the basic properties of this construction and show how to apply the new instruments in order to prove the classical topological results, such as the invariance of domain. Other similar problems we will investigate include the following:
1) Invariance of domain, general version - if U and V are homeomorphic subsets of $\mathbb{R}^n$ and U is open, also V is open.
2) Brouwer-Jordan separation theorem - if $S \subset \mathbb{R}^n$ is homeomorphic to the sphere S^{n-1}, then $\mathbb{R}^n \setminus S$ has exactly two path components and S is a boundary of both.
3) Brouwer fixed point theorem - any continuous mapping $f\colon \overline{B}^n \to \overline{B}^n$ has a fixed point i.e. $f(x) = x$ for some $x \in \overline{B}^n$.
4) S^n is not contractible to a point.
5) S^n is not a retract of $\overline{B}^{n+1}$.
6) Hairy Ball Theorem - if n is even, S^n has no non-zero tangent vector field.

In fact claims 3), 4) and 5) are equivalent - it is enough to prove one of them, since each one of them implies the others.
As an example of the converse approach - study of purely algebraic problems using topology - we will prove the Fundamental Theorem of Algebra, which says that every non-constant polynomial with complex coefficients have at least one complex root.

The singular homology theory itself is a fairly modern construction - it was invented in 1940's. All the problems listed above were already solved at that time, using earlier versions of similar ideas, such as Betti's numbers and various simplicial methods. The latter has not only historical value - simplicial methods are still very useful and important in modern mathematics, both in theory and in concrete calculations and applications. They have also given rise to a number of abstract generalizations, such as simplicial objects and related abstract combinatorial notions. That is why we start the course with a brief journey to the geometrical, concrete world of simplices and simplicial methods. This part of the course does not contain any algebra and is intended to give the reader a chance to see some concrete and more down-to-earth mathematics related to our main subjects, before diving into abstract algebra of homology theory. On the other hand concrete geometric notions of this introduction will make it easier to understand and motivate the abstract homological algebra which constitutes the main content of the course.

After some necessary algebra is developed we define and study the properties of the singular homology theory. After this machinery is complete, we apply it to topological problems such as the ones listed above. The very end of the course is dedicated to the notion of degree of a mapping $f\colon S^n \to S^n$. Historically this (and simplicial approximation) is precisely the tool Brouwer used to prove his fixed point theorem and invariance of domain theorem in the beginning of the 20th century. We define the concept of degree using the singular homology theory. Brouwer did not of course know anything about homology groups, so his definition was more complicated and geometrical in nature. However it was one of the first examples of the combinatorial invariants defined for topological objects.

In this course we only have time to scratch the very surface of the subject known as " algebraic topology ", this is why it is called "Introduction to Algebraic Topology". Perhaps the more precise name would be " Introduction to Homological Methods", since this course is mainly concerned with homology theory. Another big branches of algebraic topology include homotopy theory, K-theory, theory of obstructions and others. And of course it goes without saying that even homology theory has much more to it than what we have time for. The interested reader should start further reading from the books that are listed in the "bibliography" section.