HY Stochastic analysys, home-exam, fall 2011

December 14, 2011

Give detailed solutions of exercises:
1.b, 6 , 7, 9, 13, 14, 16, 20, 21, 22, 29

1. (a) Show that

$$(t,x) \mapsto p(t;x,y) := \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

solves the partial differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\,.$$

(b) Show that

$$(t,x) \mapsto p^{(\mu)}(t;x,y) := \frac{1}{\sqrt{2\pi t}} e^{-(x-y-\mu t)^2/2t}$$

solves

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial p}{\partial x}.$$

For which equation the function

$$(t,y) \mapsto p^{(\mu)}(t;x,y)$$

is a solution?

2. Show that for every $x > 0$

$$\frac{x}{1+x^2} e^{-x^2/2} \le \int_x^\infty e^{-u^2/2} du \le \frac{1}{x} e^{-x^2/2}.$$

3. Show that the transition density of BM on $(-\infty, a]$ reflected at a has the same form as the transition density of BM on $[a, +\infty)$ reflected at a; that is for $x, y \le a$

$$p(t;x,y) = \frac{1}{\sqrt{2\pi t}}\left(e^{-(x-y)^2/2t} + e^{-(x+y-2a)^2/2t}\right).$$

4. (a) Prove using independence of the increments that the finite dimensional distributions of B are given for $0 < t_1 < t_2 < \cdots < t_n$ by

$$\mathbf{P}_0(B_{t_1} \in A_1, B_{t_2} \in A_2, \ldots, B_{t_n} \in A_n)$$
$$= \int_{A_1} dx_1 p(t_1; 0, x_1) \int_{A_2} dx_2 p(t_2 - t_1; x_1, x_2) \cdot \ldots \cdot \int_{A_n} dx_n p(t_n - t_{n-1}; x_{n-1}, x_n)$$

(b) Prove that for $t_1 < t_2 < \cdots < t_n < t$

$$\mathbf{P}_0(B_t \in A \mid \sigma\{B_{t_1}, \ldots, B_{t_n}\}) = \int_A p(t - t_n; B_{t_n}, y) dy.$$

5. (a) Let for a fixed t

$$G_0^t := \sup\{s < t : B_s = 0\}.$$

Show that for $u < t$

$$\mathbf{P}_0(G_0^t \in du) = \frac{du}{\pi\sqrt{u(t-u)}}$$

or, equivalently,

$$\mathbf{P}_0(G_0^t < u) = \frac{2}{\pi} \arcsin \sqrt{\frac{u}{t}}, \quad u < t.$$

(b) Let

$$D_0^t := \inf\{s > t : B_s = 0\}.$$

Show that for $u \le t \le v$

$$\mathbf{P}_0(G_0^t < u, D_0^t > v) = \frac{2}{\pi} \arcsin \sqrt{\frac{u}{v}}$$

6. Consider for a given $t > 0$ the function

$$C_t := \int_0^t B_s ds.$$

(a) Explain why C_t is normally distributed.

(b) Show that $\mathbf{E}C_t = 0$ and $\mathbf{E}C_t^2 = t^3/3$ [Hint:

$$\mathbf{E}C_t^2 = \mathbf{E}\left(\left(\int_0^t B_s ds\right)^2\right) = \mathbf{E}\left(\int_0^t B_u du \int_0^t B_v dv\right) = \int_0^t \int_0^t \mathbf{E}(B_u B_v) du\, dv. \quad]$$

7. Let $B^{(i)}$, $i = 1, 2, \ldots, n$, be independent standard Brownian motions and $(x_1, \ldots, x_n) \in \mathbf{R}^n$ such that $\sum_{i=1}^n x_i^2 = 1$ set

$$Z_t := \sum_{i=1}^n x_i B_t^{(i)}.$$

Show that Z is a standard Brownian motion. Is the converse true?

8. Let B be a BM started at 0. Show that the process $\{X_t := x + \frac{l-t}{l} B\left(\frac{lt}{l-t}\right) : 0 \le t \le l\}$, where we define $X_l = x$, is a Brownian bridge of length l from x to x.

9. Let $Z_t := e^{B_t}$ be the so called geometrical BM. Find the limits

$$\lim_{h \downarrow 0} \frac{\mathbf{E}(Z_{t+h} - Z_t \mid Z_t = y)}{h}$$

and

$$\lim_{h \downarrow 0} \frac{\mathbf{E}((Z_{t+h} - Z_t)^2 \mid Z_t = y)}{h}.$$

Determine also $\mathbf{E}_y(Z_t)$ and $\mathbf{E}_y(Z_t^2)$.

10. Use optional stopping theorem to prove that for BM

$$\mathbf{P}_x(\text{hit } a \text{ before } b) = \frac{b-x}{b-a}, \quad a < x < b.$$

11. Show that for $a > 0$

$$\mathbf{P}_a(B_s \neq 0 \text{ for } 0 \le s \le t \mid B_t = a) = 1 - e^{-2a^2/t}.$$

12. Use absolute continuity to derive the distribution of the first passage time for Brownian motion with drift. Compute also the Laplace transform of the distribution.

13. Using Itô's formula show that the following are martingales

    (a) $\{B_t^3 - 3tB_t : t \ge 0\}$

    (b) $\{B_t^4 - 6tB_t^2 + 3t^2 : t \ge 0\}$.

14. Let $\tau := \inf\{t : |B_t| > a\}$, $a > 0$. Show that

    (a) $\mathbf{E}_0\tau = \mathbf{E}_0 B_\tau^2 = a^2$

    (b) $3\mathbf{E}_0\tau^2 = \mathbf{E}_0(-B_\tau^4 + 6\tau B_\tau^2) = 5a^4$.

15. Let $B^{(i)}$, $i = 1, \ldots, n$, be independent standard Brownian motions and set

$$S_t = \sum_{i=1}^{n} \left(B_t^{(i)}\right)^2.$$

Show that $\{S_t - nt : t \ge 0\}$ is a martingale and

$$< S >_t = \int_0^t 4S_r dr.$$

16. Let $B^{(i)}$, $i = 1, \dots, n$, and S be as above and set $R_t = \sqrt{S_t}$. Introduce

$$\varphi(x) = \begin{cases} \ln|x|, & n = 2 \\ |x|^{2-n}, & n \geq 3 \end{cases}$$

Show that $\{\varphi(R_t) : t \geq 0\}$ is a local martingale.

17. (a) Let $\{L_t^y : t \geq 0, y \in \mathbf{R}\}$ be the local time of the standard Brownian motion. Show that

$$\mathbf{E}_x \left(L^y_{H_a \wedge H_b} \right) = u(x, y), \quad a \leq x \leq y \leq b$$

where $H_a := \inf\{s : B_s = a\}$ and

$$u(x, y) = \frac{(x - a)(b - y)}{b - a}.$$

(b) For $a \leq y \leq x \leq b$ set $u(x, y) = u(y, x)$, and prove that

$$\mathbf{E}_x \left(\int_0^{T_a \wedge T_b} f(B_s) ds \right) = 2 \int_a^b u(x, y) f(y) dy,$$

where f is a positive, bounded, and Borel-measurable function.

18. (a) Let $S_t = \sup\{B_s : s \leq t\}$ and L_t be the local time of B at zero. Show that for $\alpha > 0$ the processes $\{(S_t - B_t + \alpha^{-1}) \exp(-\alpha S_t) : t \geq 0\}$ and $\{(|B_t| + \alpha^{-1}) \exp(-2\alpha L_t) : t \geq 0\}$ are local martingales.

(b) Let $U_x := \inf\{t : S_t - B_t > x\}$ and $T_x := \inf\{t : |B_t| > x\}$. Prove that both S_{U_x} and $2L_{T_x}$ are exponentially distributed with parameter $1/x$.

19. Let $c > 0$.

(a) Show that

$$\{(B_t, L_t^a) : t \geq 0, a \in \mathbf{R}\} \sim \{\left(\frac{1}{\sqrt{c}} B_{ct}, \frac{1}{\sqrt{c}} L_{ct}^{a/\sqrt{c}} \right) : t \geq 0, a \in \mathbf{R}\}$$

(b) Let $\tau_t := \inf\{s : L_s > t\}$. Show that

$$\{\tau_t : t \geq 0\} \sim \{\frac{1}{c} \tau_{\sqrt{c}t} : t \geq 0\}.$$

(c) Let $H_a := \inf\{s : B_s = a\}$. Show that

$$\{L_{H_a}^x : x \in \mathbf{R}, a \geq 0\} \sim \{\frac{1}{c} L_{H_{ca}}^{cx} : x \in \mathbf{R}, a \geq 0\}.$$

20. Show that the process

$$\{Z_t := \int_0^t \operatorname{sgn}(B_s) dB_s, \, t \geq 0\}$$

is a standard Brownian motion.

21. Let B be a standard linear BM. Show that $\{f(B_t) : t \geq 0\}$ is a local-$\mathcal{F}_t$-submartingale if and only if f is convex. (Recall that f is convex if $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2),\ 0 \leq \theta \leq 1$.)

22. Let f be a locally bounded Borel function on R_+ and B a standard BM. Prove that the process

$$\{Z_t := \int_0^t f(s)dB_s : t \geq 0\}$$

is Gaussian and compute its covariance $\Gamma(s,t)$. Prove that $\exp(Z_t - \frac{1}{2}\Gamma(t,t))$ is a martingale.

23. Consider the SDE

$$\begin{aligned} dN_t = &\quad rN_t dt + \alpha N_t dB_t, \\ N_0 = &\quad \xi \end{aligned}$$

Then

$$N_t = \xi e^{(r - \frac{1}{2}\alpha^2)t + \alpha B_t}$$

is the unique strong solution. Show that if ξ and $\{B_t : t \geq 0\}$ are independent then

$$\mathbf{E}N_t = e^{rt}\mathbf{E}\xi.$$

24. Solve the SDE

$$dX_t = \left(\sqrt{1 + X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1 + X_t^2}dB_t.$$

Hint: Solve first the equation

$$dX_t = \sqrt{1 + X_t^2}dB_t + \frac{1}{2}X_t dt$$

and try to use the fact that the first drift term and the diffusion term are equal.

25. Solve the SDE

$$dX_t = \left[\frac{2}{1+t}X_t - a(1+t)^2\right]dt + a(1+t)^2 dB_t.$$

26. Show that

$$X_t = \xi e^{-\alpha t} + \sigma\int_0^t e^{-\alpha(t-s)}dB_s$$

is the unique strong solution of the equation

$$\begin{aligned} dX_t &= -\alpha X_t dt + \sigma dB_s \\ X_0 &= \xi. \end{aligned}$$

Use Problem 21 to prove that

$$X \sim e^{-\alpha t} B\left(\frac{e^{2\alpha t}-1}{2\alpha}\right),$$

where it is assumed $B_0 = \xi$. (Hint: Consider the process $\{e^{\alpha t}X_t : t \geq 0\}$.)

27. Consider for a given $x \in \mathbf{R}$ the linear stochastic differential equation

$$\begin{aligned} dX_t &= A(t)X_t dt + \sigma dB_t \\ X_0 &= x. \end{aligned}$$

Assume that $A(t) \leq -\alpha < 0$ for all $t \geq 0$ and show that

$$\mathbf{E}X_t^2 \leq \frac{\sigma^2}{2\alpha} + \left(x^2 - \frac{\sigma^2}{2\alpha}\right)e^{-2\alpha t}.$$

(Hint: Use Itô's formula to find an integral equation for $s \mapsto \mathbf{E}X_s^2$. Use then the assumption on A and iterate, in other words use Gronwall's lemma: Let $t \mapsto g(t)$ be a continuous function such that for all $t > 0$

$$0 \leq g(t) \leq \alpha(t) + \beta \int_0^t g(s)ds$$

with $\beta \geq 0$ and $t \mapsto \alpha(t)$ integrable. Then

$$g(t) \leq \alpha(t) + \beta \int_0^t \alpha(s)e^{\beta(t-s)}ds.\ )$$

28. Consider the system

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= -\beta X_t dt - \alpha Y_t dt + \sigma dB_t \end{aligned}$$

where α, β, σ are positive constants.

(a) Solve the system.

(b) Show that if (X_0, Y_0) has a Gaussian distribution then (X_t, Y_t) is a time homogeneous Gaussian process.

(c) Find the covariance function of this process.

29. (a) Prove that for $t \in [0, 1)$ and $x \in \mathbf{R}$ the solution to the SDE

$$X_t^x = B_t + \int_0^t \frac{x - X_s^x}{1-s}ds \qquad (\dagger)$$

is given by

$$X_t^x = xt + B_t - (1-t)\int_0^t \frac{B_s ds}{(1-s)^2} = xt + (1-t)\int_0^t \frac{dB_s}{1-s}.$$

(b) Prove that

$$\lim_{t \uparrow 1} X_t^x = x \quad \text{a.s.}$$

and that, if we set $X_1^x = x$, then X_t^x, $t \in [0,1]$, is a Brownian bridge, that is $\{X_t^x : 0 \le t \le 1\}$ is a Gaussian process such that $\mathbf{E}X_t^x = xt$, and for $x = 0$, $\mathbf{E}(X_s^x X_t^x) = s(1-t)$, $s \le t \le 1$.

Hint: to prove (†) use the following result: Suppose that a) g is decreasing and continuous on $[0,1]$ and $g(1) = 0$, b) f is positive on $[0,1)$, c) $\int_0^1 f(x)g(x)dx < \infty$ then

$$\lim_{t \to 1-} g(t) \int_0^t f(s)ds = 0.$$